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SOME ALGEBRAIC SETS OF HIGH LOCAL COHOMOLOGICAL DIMENSION IN PROJECTIVE SPACE

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ABSTRACT. Let $V_0, \ldots, V_{[n/t]}$ be algebraic sets of pure codimension t in P^n, and suppose $\bigcap V_i$ is empty. Then $P^n - \bigcup V_i$ has cohomological dimension $n - [n/t]$.

If U is a scheme, then $\mathrm{cd}(U)$, the cohomological dimension of U, is the largest integer i such that there exists a quasi-coherent sheaf F on U such that $H^i(F) \neq 0$.

In [1], G. Faltings proved that if V is an algebraic set of pure codimension t in P^n, then

$$\mathrm{cd}(P^n - V) \leqslant n - [n/t]. \tag{1}$$

This note gives some algebraic sets for which equality holds in (1).

THEOREM. *Put $s = [n/t]$ and let $V = V_0 \cup V_1 \cup \cdots \cup V_s$ be the union of $s + 1$ algebraic sets of pure codimension t in general position in P^n (i.e. such that the intersection of all of them is empty). Then*

$$\mathrm{cd}(P^n - V) = n - [n/t].$$

This theorem (from the author's thesis [4]) answers the conjecture from [3] in the affirmative and covers all three examples from [3], but not the statement of the main theorem.

For a proof it is convenient to translate the problem into an algebraic language. Put $R_n = k[x_0, \ldots, x_n]_{(x_0, \ldots, x_n)}$ and let $\mathfrak{A}$ be the defining ideal of V in R_n. Then the cohomological dimension of $P^n - V$ is the largest integer i such that $H^{i+1}_{\mathfrak{A}}(R_n) \neq 0$ (cf. [2]).

LEMMA. *Put $s = [n/t]$ and let $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_j$ be $j + 1$ homogeneous ideals of pure height t in R_n. Put $\beta_j = \sum_{r=0}^{r=j} \mathfrak{A}_r$. Then $H^i_{\beta_j}(R_n) = 0$ if $i \geqslant n - s + j + 2$.*

PROOF. If $j = 0$, the result follows from (1). Put $\beta_{j-1} = \sum_{r=0}^{r=j-1} \mathfrak{A}_r$. Then $\beta_{j-1} \cap \mathfrak{A}_j$ has the same radical as $\gamma_{j-1} = \sum_{r=0}^{r=j-1} (\mathfrak{A}_r \cap \mathfrak{A}_j)$. Since β_{j-1} and γ_{j-1} are sums of $j - 1$ ideals of pure heights t in R_n, we may assume that $H^i_{\beta_{j-1}}(R_n) = H^i_{\gamma_{j-1}}(R_n) = 0$ for all $i \geqslant n - s + j + 1$. We also know that $H^i_{\mathfrak{A}_j}(R_n) = 0$ if $i \geqslant n - s + 2$. The Mayer-Vietoris long exact sequence gives

$$H^i_{\gamma_{j-1}}(R_n) \to H^{i+1}_{\beta_j}(R_n) \to H^{i+1}_{\beta_{j-1}}(R_n) \oplus H^{i+1}_{\mathfrak{A}_j}(R_n)$$

and this proves the Lemma.

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PROOF OF THE THEOREM. Let $\mathfrak{A}_0, \ldots, \mathfrak{A}_s$ be the defining ideals of $V_0, \ldots, V_s$ in R_n. Put $\mathfrak{S}_j = \mathfrak{A}_0 \cap \mathfrak{A}_1 \cap \cdots \cap \mathfrak{A}_j + \mathfrak{A}_{j+1} + \cdots + \mathfrak{A}_s$. Then the biggest integer i for which $H^i_{\mathfrak{S}_s}(R_n) \neq 0$ is $i = n - j + 1$. We are going to prove this by induction on j and the theorem will follow by putting $j = s$.

For $j = 0$, $\mathfrak{S}_j$ is m-primary, where m is the maximal ideal of R_n and the above claim is well known in this case. Assume $j > 0$ and assume the Theorem proven for $j - 1$. Put $\mathfrak{A}' = \mathfrak{A}_j + \mathfrak{A}_{j+1} + \cdots + \mathfrak{A}_s$ and $\mathfrak{A}'' = \mathfrak{A}_0 \cap \mathfrak{A}_1 \cap \cdots \cap \mathfrak{A}_{j-1} + \mathfrak{A}_{j+1} + \mathfrak{A}_{j+2} + \cdots + \mathfrak{A}_s$. Then $\mathfrak{S}_j = \mathfrak{A}' \cap \mathfrak{A}''$ and $\mathfrak{S}_{j-1} = \mathfrak{A}' + \mathfrak{A}''$. By the Lemma $H^i_{\mathfrak{A}'}(R_n) = H^i_{\mathfrak{A}''}(R_n) = 0$ for all $i \geqslant n - j + 2$. The claim now follows from the Mayer-Vietoris sequence considering the induction hypothesis. Q.E.D.

REMARK. The above Lemma gives a lower bound on the number of algebraic sets of given codimension which are needed to cut out a given algebraic subset of $\mathbf{P}^n$ set-theoretically. Namely, if $V \subset \mathbf{P}^n$ and $\mathrm{cd}(\mathbf{P}^n - V) = v$, then we need at least $v + 1 - (n - [n/t])$ algebraic subsets of pure codimension t to cut out V set-theoretically.

Faltings' inequality (1) and the fact that it is exact for all n and t (our Theorem) suggest the following.

CONJECTURE. Every algebraic subset of $\mathbf{P}^n$ of pure codimension t is a set-theoretic intersection of $n + 1 - [n/t]$ hypersurfaces [**4**, p. 8].

For additional supporting evidence for this conjecture see [**4**, Theorem 6].

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