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EXPONENTIAL SUMS OF LERCH'S ZETA FUNCTIONS

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ABSTRACT. For x not an an integer and $\operatorname{Re}(s) > 0$, let

$$F(x,s) = \sum_{k=1}^{\infty} \frac{e^{2\pi i k x}}{k^s}$$

be the Lerch's zeta function. In this note, we will show that

$$\sum_{\gamma=1}^{m-1} e^{-2\pi i \gamma \alpha/m} F\left(\frac{\gamma}{m}, 1-n\right) = \frac{1}{n}\left(B_n - m^n B_n\left(\frac{\alpha}{m} - \left[\frac{\alpha}{m}\right]\right)\right)$$

where α is an integer and $\alpha \not\equiv 0 \pmod{m}$ and $n \geq 1$. For $n = 1$, this formula is equivalent to the classical Eisenstein formula

$$\frac{\alpha}{m} - \left[\frac{\alpha}{m}\right] - \frac{1}{2} = -\frac{1}{2m} \sum_{\gamma=1}^{m-1} \sin\frac{2\pi\gamma\alpha}{m} \cot\frac{\pi\gamma}{n}.$$

1. Introduction. About fifteen years ago, S. Chowla [**3**] proved the following interesting result:

THEOREM A. *Let p be a prime. Then the $\frac{1}{2}(p-1)$ real numbers $\cot \gamma\pi/p$, $\gamma = 1, \ldots, \frac{1}{2}(p-1)$, are linear independent over Q, the field of rational numbers.*

Recently, we have extended this result to nonprime numbers and the derivatives of all orders of $\cot x$ [**5**].

THEOREM B. *For an arbitrary positive integer m, let $\Lambda = \{\gamma | 1 \leq \gamma < \frac{1}{2}m, (\gamma, m) = 1\}$. Then for each $s = 0, 1, \ldots,$ the $\frac{1}{2}\phi(m)$ real numbers*

$$\left\{ \left(\frac{d^s}{dx^s} \cot\frac{(x+\gamma)\pi}{m} \right)_{x=0} \middle| \gamma \in \Lambda \right\}$$

are linear independent over Q where $\phi(m)$ is the Euler function.

On the other hand, there is a classical Eisenstein formula

$$\frac{\alpha}{m} - \left[\frac{\alpha}{m}\right] - \frac{1}{2} = -\frac{1}{2m} \sum_{\gamma=1}^{m-1} \sin\frac{2\pi\gamma\alpha}{m} \cot\frac{\pi\gamma}{m}.$$

Our recent result suggests that it may be of interest to extend the Eisenstein formula to the derivatives of all orders of $\cot x$. Our main result is the following theorem.

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THEOREM C. *Let m be an arbitrary positive integer. For $s = 1$, let $E_s = \frac{1}{2}$ and for $s \neq 1$, let $E_s = 0$. If s is odd and $s \geq 1$, then*

$$\sum_{\alpha=0}^{m-1} \sin \frac{2\pi\alpha\gamma}{m} \left(\frac{d^{s-1}}{dx^{s-1}} \cot \left(x + \frac{\alpha\pi}{m} \right) \right)_{x=0} = \frac{2^s i^{s-1}}{s} \left(m^s B_s \left(\frac{\gamma}{m} - \left[\frac{\gamma}{m} \right] \right) - B_s + E_s \right).$$

If s is even and $s \geq 1$, then

$$\sum_{\alpha=0}^{m-1} \cos \frac{2\pi\alpha\gamma}{m} \left(\frac{d^{s-1}}{dx^{s-1}} \cot \left(x + \frac{\alpha\pi}{m} \right) \right)_{x=0} = \frac{(2i)^s}{s} (m^s B_s \left(\frac{\gamma}{m} - \left[\frac{\gamma}{m} \right] \right) - B_s,$$

where $\gamma \not\equiv 0 \pmod{m}$, B_s and $B_s(x)$ are Bernoulli numbers and Bernoulli polynomials, respectively.

We refer to §2 for the definitions of Bernoulli polynomials and Bernoulli numbers.

Theorem C will be proved as a corollary to Theorem D in §2 which involves Lerch's zeta functions.

2. Lerch's zeta functions. Let $\phi(z, s)$ be the power Dirichlet series defined by

$$\phi(z, s) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

which converges for all s if $|z| < 1$, for $\mathrm{Re}(s) > 0$ if $|z| = 1$, $z \neq 1$, and for $\mathrm{Re}(s) > 1$ if $z = 1$. It is known that $\phi(z, s)$ can be extended to the whole s-plane by means of the contour integral. Let $z = e^{2\pi i x}$, where x is real, and the periodic function $F(x, s)$ is defined by the equation $F(x, s) = \phi(e^{2\pi i x}, s)$. $F(x, s)$ is called Lerch's zeta function in the literature. For x not an integer and $\mathrm{Re}(s) > 0$, we have the series representation

$$F(x, s) = \sum_{k=1}^{\infty} \frac{e^{2\pi i k x}}{k^s}.$$

On the other hand, let $B_n(x)$ be the nth Bernoulli polynomials, defined by the power series

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

and $B_n = B_n(0)$ is the nth Bernoulli number.

In this note we will prove the following result:

THEOREM D. *Let $m > 2$ be a fixed positive integer and let α be an integer such that $\alpha \not\equiv 0 \pmod{m}$. Then, for $n \geq 1$,*

$$\sum_{\gamma=1}^{m-1} e^{-2\pi i \gamma \alpha / m} F \left(\frac{\gamma}{m}, 1 - n \right) = \frac{1}{n} \left(B_n - m^n B_n \left(\frac{\alpha}{m} - \left[\frac{\alpha}{m} \right] \right) \right).$$

Our proof involves the Fourier series for the Bernoulli polynomials. This was suggested by a referee of a previous version of this note.

3. A proof of Theorem D. Let $B_n(x)$ be the nth Bernoulli polynomial as defined in the Introduction. Then $B_n(x-[x])$ is a periodic function of period 1. It is known that the Fourier series of $B_n(x-[x])$ are given as follows [**4**, p. 16]:

$$B_{2k-1}(x-[x]) = 2(-1)^k(2k-1)!\sum_{h=1}^{\infty}\frac{\sin 2\pi hx}{(2\pi h)^{2k-1}}$$

and

$$B_{2k}(x-[x]) = 2(-1)^{k-1}(2k)!\sum_{h=1}^{\infty}\frac{\cos 2\pi hx}{(2\pi h)^{2k}}$$

where $B_1(x-[x])$ has to be exempted for integral x because the discontinuity of $B_1(x-[x])$ at x is equal to 0. For our purpose, we may combine the above two formulas into one;

$$B_n(x-[x]) = -(-i)^n n!\sum_{h=1}^{\infty}\frac{e^{2\pi ihx}+(-1)^n e^{-2\pi ihx}}{(2\pi h)^n}$$

for $n>1$.

Recall that $F(x,s)$ satisfies the following functional equation;

$$F(x,s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}(e^{\pi i(1-s)/2}\xi(1-s,x)+e^{\pi i(s-1)/2}\xi(1-s,1-x)),$$

where $\Gamma(s)$ is the gamma function, $\xi(s,x)$ is the Hurwitz zeta function.

Recall that for $0<x\le 1$ and $\mathrm{Re}(s)>1$,

$$\xi(s,x) = \sum_{h=0}^{\infty}\frac{1}{(h+x)^s}.$$

Since for $n=1$, Theorem D is equivalent to Eisenstein's formula, we will only consider the case $n>1$:

$$\begin{aligned}
&\sum_{\gamma=1}^{m-1} e^{-2\pi i\gamma\alpha/m}F\left(\frac{\gamma}{m},1-n\right)\\
&= \frac{(n-1)!i^n}{(2\pi)^n}\left(\sum_{\gamma=1}^{m-1} e^{-2\pi i\gamma\alpha/m}\xi\left(n,\frac{\gamma}{m}\right)+(-1)^n\sum_{\gamma=1}^{m-1} e^{-2\pi i\gamma\alpha/m}\xi\left(n,\frac{m-\gamma}{m}\right)\right)\\
&= \frac{(n-1)!i^n}{(2\pi)^n}\left(\sum_{\gamma=1}^{m-1} e^{-2\pi\gamma\alpha/m}\xi\left(n,\frac{\gamma}{m}\right)+(-1)^n\sum_{\beta=1}^{m-1} e^{2\pi i\beta\alpha/m}\xi\left(n,\frac{\beta}{m}\right)\right)\\
&= \frac{(n-1)!i^n}{(2\pi)^n}\left(\sum_{\gamma=1}^{m-1}\sum_{k=0}^{\infty}\frac{e^{2\pi i\gamma\alpha/m}+(-1)^n e^{-2\pi i\gamma\alpha/m}}{(k+\gamma/m)^n}\right)
\end{aligned}$$

(and letting $h = mk + \gamma$)

$$= m^n (n-1)! i^n \sum_{h=1, h \not\equiv O(m)}^{\infty} \frac{e^{2\pi i h\alpha/m} + (-1)^n e^{-2\pi i h\alpha/m}}{(2\pi h)^n}$$

$$= m^n (n-1)! i^n \left(\sum_{h=1}^{\infty} \frac{e^{2\pi i h\alpha/m} + (-1)^n e^{-2\pi i h\alpha/m}}{(2\pi h)^n} \right.$$

$$\left. - \sum_{k=1}^{\infty} \frac{e^{2\pi i \alpha k} + (-1)^n e^{-2\pi i \alpha k}}{(2\pi m k)^n} \right)$$

$$= m^n \left(-\frac{1}{n} B_n \left(\frac{\alpha}{m} - \left[\frac{\alpha}{m} \right] \right) + \frac{1}{n m^n} B_n(\alpha - [\alpha]) \right)$$

$$= \frac{1}{n} \left(B_n - m^n B_n \left(\frac{\alpha}{m} - \left[\frac{\alpha}{m} \right] \right) \right).$$

This completes the proof.

4. Examples. The first few values of $F(x, -n)$ are given in [**2**] as follows:

$$F(x,0) = -\tfrac{1}{2}(1 - i \cot \pi x), \quad F(x,-1) = -\tfrac{1}{4}(1 + \cot^2 \pi x),$$
$$F(x,-2) = -\tfrac{i}{8}(2 \cot \pi x + 2 \cot^3 \pi x).$$

It follows that

$$\sum_{\alpha=1}^{m-1} \cos \frac{2\pi\alpha\gamma}{m} \cot^2 \frac{\alpha\pi}{m} = \frac{2}{3} + 2m^2 B_2 \left(\frac{\gamma}{m} - \left[\frac{\gamma}{m} \right] \right),$$

$$\sum_{\alpha=1}^{m-1} \sin \frac{2\pi\alpha\gamma}{m} \cot^3 \frac{\alpha\pi}{m} = -2m B_1 \left(\frac{\gamma}{m} - \left[\frac{\gamma}{m} \right] \right) + \frac{4m^3}{3} B_3 \left(\frac{\gamma}{m} - \left[\frac{\gamma}{m} \right] \right).$$

5. A proof of Theorem C. Now Theorem C follows easily from Theorem D because

$$F(x,0) = \tfrac{1}{2}(1 - i \cot \pi x).$$

6. Comments. (1) Theorem C is equivalent to the following:

THEOREM E. *Let $m > 2$ be a fixed positive integer and let α be an integer such that $\alpha \not\equiv 0 \pmod{m}$. Then,*

$$\frac{1}{2i} \sum_{\gamma=1}^{m-1} e^{-2\pi i \alpha\gamma/m} \cot \left(x + \frac{\gamma\pi}{m} \right)$$
$$= \frac{1}{2} - \frac{e^{2xi}}{e^{2xi} - 1} + \frac{m e^{2mxi} (\gamma/m - [\gamma/m])}{e^{2mxi} - 1}.$$

This formula was originally proved by different arguments without involving Lerch's zeta function. The computation is considerably longer.

(2) Theorem D can also be used to derive the formula for Dirichlet L-function using the following representation of Dirichlet L-function for primitive mod m Dirichlet character χ [**2**]:

$$\tau(\overline{\chi}) L(s, \chi) = \sum_{\alpha=1}^{m-1} \overline{\chi}(\alpha) F\left(\frac{\alpha}{m}, s \right),$$

where $\tau(\overline{\chi})$ is the Gauss sum.

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