A CHARACTERIZATION OF POTENTIAL SPACES

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ABSTRACT. A mean oscillation characterization, valid for all $\alpha > 0$, of the spaces $L^p_\alpha$ of Bessel potentials of $L^p$ functions is given and is used to relate the known characterizations for $0 < \alpha < 2$ via Marcinkiewicz integrals, due to E. M. Stein, and via vector-valued means of differences, due to R. S. Strichartz.

1. Introduction. The Bessel potential of order $\alpha$, $\alpha > 0$, of a smooth function $g$ is defined as $J_\alpha g = K_\alpha * g$, where

$$ (K_\alpha)\hat{\gamma}(\xi) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}. $$

The kernel $K_\alpha$ can be shown to be integrable and, for $1 \leq p \leq \infty$, the potential spaces $L^p_\alpha$, introduced by Aronszajn and Smith and by Calderón, are then the spaces of Bessel potentials of order $\alpha$ of $L^p$ functions: that is,

$$ L^p_\alpha = \{J_\alpha g: g \in L^p\}. $$

With the norm $\|J_\alpha g\|_{p,\alpha} = \|g\|_p$, $L^p_\alpha$ becomes a Banach space.

This paper is concerned with the question of when a given function $f$ is the Bessel potential of an $L^p$ function. An obviously necessary condition is that $f$ be in $L^p$, and Stein proved [7], for $0 < \alpha < 2$ and $2n/n + 2\alpha < p < \infty$, that an $L^p$ function $f$ belongs to $L^p_\alpha$ if and only if its Marcinkiewicz integral

$$ D_\alpha f(x) = \left(\int_{\mathbb{R}^n} |\Delta_y^{[\alpha]+1} f(x)|^2 |y|^{-2\alpha-n} dy\right)^{1/2} $$

is in $L^p$ and, furthermore, $\|f\|_{p,\alpha} \sim \|f\|_p + \|D_\alpha f\|_p$ (by $A \sim B$ we mean there is a constant $C$ independent of $A$ or $B$ such that $C^{-1}A \leq B \leq CA$; also, $\Delta_y f(x) = f(x+y) - f(x)$).

With $p$ now such that $1 < p < \infty$, the following characterization of Bessel potentials for the same range of $\alpha$ is due to Strichartz [9]. Let $[\alpha]$ denote the integral part of $\alpha$.

THEOREM 1. Let $0 < \alpha < 2$ and $1 < p < \infty$. An $L^p$ function $f$ belongs to $L^p_\alpha$ if and only if the function

$$ S_{\alpha} f(x) = \left(\int_0^\infty \left(t^{-\alpha} \int_{|y| \leq 1} |\Delta_y^{[\alpha]+1} f(x)| dt\right)^2 t^{-1} dt\right)^{1/2} $$

belongs to $L^p$. Furthermore, $\|f\|_{p,\alpha} \sim \|f\|_p + \|S_{\alpha} f\|_p$.

This result was extended by Bagby [1] to the range $0 < \alpha < n$. 

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We will show here how Bessel potentials of $L^p$ functions, $1 < p < \infty$, can be characterized for a general $\alpha > 0$. This will be done by means of a mixed norm estimate of the approximation of these functions by polynomials. To be more precise, let $f$ be a locally integrable function and $Q$ a cube in $\mathbb{R}^n$. We denote by $P_Q^k f$ the unique polynomial of degree $k$ such that

$$
\int_Q (f(y) - P_Q^k f(y)) y^\gamma dy = 0
$$

for each $n$-tuple $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$ such that $|\gamma| = \gamma_1 + \cdots + \gamma_n \leq k$ (for instance, if $k = 0$, $P_Q^0 f$ is just the mean $f_Q = |Q|^{-1} \int_Q f$). We will write $P_Q f$ or even $P_Q$ if there is no chance of confusion. Then, for $x \in \mathbb{R}^n$ and $t > 0$, we define

$$
\Omega^k_f(x, t) = \Omega_f(x, t) = \sup \left\{ |Q|^{-1} \int_Q |f - P_Q^k f| : x \in Q, |Q| = t^n \right\}.
$$

The main result of this paper is

**Theorem 2.** Let $\alpha > 0$ and $f \in L^p$, $1 < p < \infty$. Then $f \in L^p_\alpha$ if and only if the function

$$
G_\alpha f(x) = \left( \int_0^{\infty} (t^{-\alpha} \Omega_f^{|\alpha|}(x, t))^2 t^{-1} dt \right)^{1/2}
$$

is in $L^p$. Furthermore, $\|f\|_{p, \alpha} \sim \|f\|_p + \|G_\alpha f\|_p$.

The paper is organized as follows: §2 contains the proof of Theorem 2 when $0 < \alpha < 1$; in view of Strichartz’s theorem, we will prove that $G_\alpha f \in L^p$ if and only if $S_\alpha f \in L^p$. In §3 we deal with the case $\alpha > 0$, $\alpha$ nonintegral, by reducing it to that of §2. Complex interpolation is then used in §4 to deal with the remaining case $\alpha > 0$, $\alpha$ integral. Finally, §5 contains the relationship between $G_\alpha f$ and the Marcinkiewicz integral $D_\alpha f$, and some further results.

**2. The case $0 < \alpha < 1$.** We begin with some remarks about the polynomials $P_Q$. Let $f$ be a locally integrable function. An easy homogeneity argument gives [6]

$$
\text{ess sup}_Q |P_Q^k f| \leq C |Q|^{-1} \int_Q |f|,
$$

for any cube $Q$ and $k \geq 0$, and where $C$ does not depend on $Q$ or $f$ (throughout the paper $C$ will denote any absolute constant independent of the particular functions, points or sets considered). As a consequence, for any $n$-tuple $\gamma \in \mathbb{N}^n$, we have for the $\gamma$-derivatives of $P_Q$ (see [3, 4])

$$
\text{ess sup}_Q |D^\gamma (P_Q^k f)| \leq C |Q|^{-\gamma/n} \text{ess sup}_Q |P_Q^k f|
$$

$$
\leq C |Q|^{-\gamma/n - 1} \int_Q |f|.
$$

It also follows from (1) that, since $P_Q^k (f + R) = P_Q^k f + R$ for any polynomial $R$ of degree $\leq k$,

$$
|Q|^{-1} \int_Q |f - P_Q^k f| \leq C |Q|^{-1} \int_Q |f - R|,
$$

for any cube $Q$ and polynomial $R$ of degree $\leq k$. The proof of Theorem 2 then proceeds as follows. Assume $f \in L^p_\alpha$, and let $\Omega_f(x, t) = \Omega^0_f(x, t)$. For $t > 0$ and $R$ a polynomial of degree $\leq k$ of mean $0$ on $Q$, we have

$$
\Omega_f(x, t) \leq C (t \Omega_f(x, t))^{1/2} \leq C \left( \int_0^{\infty} (t^{-\alpha} \Omega_f^{|\alpha|}(x, t))^2 t^{-1} dt \right)^{1/2} \leq C \left( \int_0^{\infty} \int_Q (t^{-\alpha} |f - P_Q^k f|^2)^2 t^{-1} dt \right)^{1/2} \leq C \left( \int_0^{\infty} \int_Q (t^{-\alpha} |f - R|^2)^2 t^{-1} dt \right)^{1/2}.
$$

Since $(t^{-\alpha} |f - R|^2) \leq \frac{2}{t^\alpha}$, we have

$$
\Omega_f(x, t) \leq 2 C \left( \int_0^{\infty} (t^{-\alpha} |f - R|^2)^2 t^{-1} dt \right)^{1/2} \leq C \left( \int_0^{\infty} (t^{-\alpha} |f|^2)^2 t^{-1} dt \right)^{1/2} \leq C \left( \int_0^{\infty} (t^{-\alpha} |f|^2)^2 t^{-1} dt \right)^{1/2}.
$$

Since $\|f - R\|_{L^p_\alpha} \leq \|f - R\|_{L^p}$, we have $f \in L^p_\alpha$.

For $\alpha > 1$, we use the inequality $\|f - R\|_{L^p} \leq C \left( \int_0^{\infty} (t^{-\alpha} |f|^2)^2 t^{-1} dt \right)^{1/2}$ and the fact that $\Omega_f(x, t) \leq C \left( \int_0^{\infty} (t^{-\alpha} |f|^2)^2 t^{-1} dt \right)^{1/2}$ for any $t > 0$ and polynomial $R$ of degree $\leq k$. Then $\Omega_f(x, t) \leq C \left( \int_0^{\infty} (t^{-\alpha} |f|^2)^2 t^{-1} dt \right)^{1/2} \leq C \left( \int_0^{\infty} (t^{-\alpha} |f|^2)^2 t^{-1} dt \right)^{1/2}$. Since $\|f - R\|_{L^p} \leq \|f - R\|_{L^p}$, we have $f \in L^p_\alpha$. This completes the proof of Theorem 2.
and that if \( Q \subset Q' \),

\[
|Q|^{-1} \int_Q |f - P^k_Q f| \leq C(|Q'|/|Q|)|Q'|^{-1} \int_{Q'} |f - P^k_{Q'} f|.
\]

The cube with centre \( x \) and side length \( t \) will be denoted by \( Q_{x,t} \). From (4) we obtain

\[
\Omega^k_f(x,t) \leq C t^{-n} \int_{Q_{x,2t}} |f - P_{Q_{x,2t}}|,
\]

and therefore, if \( t \leq s \leq 2t \), \( \Omega^k_f(x,t) \leq C \Omega^k_f(x,s) \leq C \Omega^k_f(x,2t) \). In particular, the series

\[
\left( \sum_{i=0}^{\infty} (2^{-i} \Omega_f(x,2^i))^2 \right)^{1/2}
\]

is comparable to \( G_\alpha \), and we can also use balls instead of cubes to define \( \Omega_f \) and \( G_\alpha f \). We have now

**Theorem 3.** Let \( \alpha > 0 \), \( \alpha \) nonintegral, and \( f \in L^p \), \( 1 < p < \infty \). Then \( G_\alpha f \in L^p \) if and only if for a.e. \( x \in \mathbb{R}^n \) there is a polynomial \( P_x \) of degree \( \leq [\alpha] \), depending on \( f \), such that if

\[
\tilde{G}_\alpha f(x) = \left( \int_0^\infty \left( t^{-\alpha-n} \int_{Q_{x,t}} |f(y) - P_x(y)| dy \right)^2 t^{-1} dt \right)^{1/2},
\]

\( \tilde{G}_\alpha f \in L^p \). Furthermore, \( \|G_\alpha f\|_p \sim \|\tilde{G}_\alpha f\|_p \). The polynomial \( P_x \) will be a.e. the Taylor polynomial of \( f \).

**Proof.** The inequality \( G_\alpha f \leq C \tilde{G}_\alpha f \) is an immediate consequence of (5) and (3). Conversely, if \( G_\alpha f \in L^p \), fix \( x \in \mathbb{R}^n \), \( \gamma \in \mathbb{N}^n \) with \( |\gamma| \leq [\alpha] \) and write

\[
P_{Q_{x,t},\gamma} f(y) = P_{Q_{x,t}}(y) = \sum_{|\beta| \leq [\alpha]} a_\beta(x,t)(y-x)^\beta/\beta!.
\]

If \( t > s > 0 \), let \( Q_0 \subset Q_1 \subset \cdots \subset Q_m \) be a sequence of cubes such that \( Q_0 = Q_{x,s}, Q_m = Q_{x,t} \) and \( |Q_i| = 2^n |Q_{i-1}| \). Then, by (2),

\[
|a_\gamma(x,t) - a_\gamma(x,s)| = |D^\gamma(P_{Q_0} - P_{Q_m})(x)|
\]

\[
\leq \sum_1^m |D^\gamma(P_{Q_i} - P_{Q_{i-1}})(x)|
\]

\[
\leq C \sum_1^m |Q_i|^{-1-|\gamma|/n} \int_{Q_i} |f - P_{Q_i}|
\]

\[
\leq C \int_s^t \Omega_f(x,u)u^{-|\gamma|-1} du.
\]

Thus, if \( \alpha > |\gamma| \), Schwartz’s inequality gives

\[
|a_\gamma(x,t) - a_\gamma(x,s)| \leq Ct^{\alpha-|\gamma|} G_\alpha f(x).
\]
In addition, we have

\[ |a_\gamma(x,t)| \leq |a_\gamma(x,t) - a_\gamma(x,1)| + |a_\gamma(x,1)| \]
\[ \leq C(G_\alpha f(x) + Mf(x)), \]

with \( M \) the Hardy-Littlewood maximal operator. Therefore, as \( t \) goes to 0, the functions \( a_\gamma(x,t) \) form a Cauchy sequence of \( L^p \) functions. We denote its \( L^p \) limit by \( f_\gamma \), which will also be a.e. the pointwise limit of some subsequence \( a_\gamma(x,t_j) \).

Hence, for a.e. \( x \)

\[ |f_\gamma(x) - a_\gamma(x,t)| = \lim |a_\gamma(x,t_j) - a_\gamma(x,t)| \]
\[ \leq C \int_0^t \Omega_f(x,u) u^{-|\gamma|-1} \, du \leq Ct^{\alpha-|\gamma|} G_\alpha f(x), \]

and, in particular, \( \|f_\gamma\|_p \leq \|f_\gamma - a_\gamma(\cdot,1)\|_p + \|a_\gamma(\cdot,1)\|_p \leq C(\|G_\alpha f\|_p + \|f\|_p) \).

If we now define \( P^\gamma(x) = \sum_{|\gamma| \leq k} f_\gamma(y - x)^\gamma/\gamma! \), we obtain

\[
\int_{Q_{x,t}} |f - P^\gamma x| \, dy \leq \int_{Q_{x,t}} |f - P_{Q_{x,t}}| \, dy + C \sum_{|\gamma| \leq k} |a_\gamma(x,t) - f_\gamma(x)| t^{|\gamma|}
\]
\[ \leq Ct^{-n} \left( \Omega_f(x,t) + \sum_{i=0}^k t^i \int_0^t \Omega_f(x,u) u^{-i-1} \, du \right). \]

Assuming that \( k = [\alpha] < \alpha \), Hardy’s inequality implies

\[ \tilde{G}_\alpha f(x) \leq CG_\alpha f(x) \quad \text{for a.e. } x \in \mathbb{R}^n. \]

In particular, since \( a_0(x,t) = f_{Q_{x,t}} \) tends to \( f(x) \) a.e., it follows that for \( 0 < \alpha < 1 \), \( \tilde{G}_\alpha f(x) = S_\alpha f(x) \) a.e., and Theorem 2 is proved in this case.

3. The case \( \alpha > 0, \alpha \) nonintegral. The following is a well-known result in the theory of potential spaces (see, for instance [8, p. 136]).

**Theorem 4.** Let \( f \in L^p, 1 < p < \infty \). Then \( f \in L^p_\alpha \), \( \alpha \geq 1 \), if and only if \( f \) and its weak partials \( \partial f/\partial x_i \) belong to \( L^p_{\alpha - 1} \). Furthermore, \( \|f\|_{p,\alpha} \sim \|f\|_{p,\alpha - 1} + \sum_1^n \|\partial f/\partial x_i\|_{p,\alpha - 1} \).

A similar result also holds for \( L^p \) functions \( f \) such that \( G_\alpha f \in L^p \); concretely,

**Theorem 5.** Let \( f \in L^p, 1 < p < \infty \). Then, for \( \alpha > 1 \), \( G_\alpha f \in L^p \) if and only if \( G_{\alpha - 1} f \in L^p \), the weak partials \( \partial f/\partial x_i \in L^p \) and \( G_{\alpha - 1}(\partial f/\partial x_i) \in L^p \). Furthermore,

\[ \|f\|_p + \|G_\alpha f\|_p \sim \|f\|_p + \|G_{\alpha - 1} f\|_p + \sum_1^n (\|\partial f/\partial x_i\|_p + \|G_{\alpha - 1}(\partial f/\partial x_i)\|_p). \]

**Proof.** We begin with the only if part. Suppose first that \( f \) is a \( C^1 \) function such that \( f \) and \( \partial f/\partial x_i \) are in \( L^p \), and fix \( x \) and \( t > 0 \). For each \( i = 1, \ldots, n \) and \( y \in Q_{0,t} \), consider the polynomials

\[ P_i(x + y) = \int_0^1 P_{Q_{x,t}}^{k-1}(\partial f/\partial x_i)(x + sy) \, ds, \]
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with \( k = [\alpha] \). Then, by Taylor’s formula,

\[
t^{-n} \int_{Q_{x,t}} \left| f(z) - f(x) - \sum_{i=1}^{n} P_i(z)(z_i - x_i) \right| dz
\]

\[
\leq t^{-n} \sum_{i=1}^{n} \int_{Q_{o,t}} \left| \partial f/\partial x_i(x + sy) - P_{Q_{x,t}}^{k-1}(\partial f/\partial x_i)(x + sy) \right| dy ds
\]

\[
\leq C \sum_{i=1}^{n} t \int_{0}^{1} (st)^{-n} \int_{Q_{o,t}} \left| \partial f/\partial x_i(x + z) - P_{Q_{x,t}}^{k-1}(x + z) \right| dz ds,
\]

which implies

\[
\Omega^k_f(x,t) \leq C \sum_{i=1}^{n} \int_{0}^{2t} \Omega_{\partial f/\partial x_i}^{k-1}(x,u) du.
\]

This estimate can be extended to any \( f \in L^p \) with weak partials in \( L^p \) by a standard regularization argument and, in the conditions of the theorem, Hardy’s inequality yields

\[
G_{\alpha}f(x) \leq C \sum_{i=1}^{n} G_{\alpha-1}(\partial f/\partial x_i)(x).
\]

Next, we will divide the proof of the if part into three lemmas.

**Lemma 1.** Let \( f \in L^p \) such that \( G_{\alpha}f \in L^p \), and \( \gamma \in \mathbb{N}^n \), \( |\gamma| < \alpha \). Then, with \( f_\gamma \) as in Theorem 3, \( \|G_{\alpha-|\gamma|}f_\gamma\|_p \leq C\|G_{\alpha}f\|_p \).

**Proof.** Fix \( x \in \mathbb{R}^n \) and a cube \( Q \) with \( x \in Q \) and \( |Q| = t^n \). If \( y \in Q \), \( Q_{y,t} \subset Q_{x,2t} \), and by (8) and (2),

\[
|f_\gamma(y) - D^\gamma P_{Q_{x,2t}}(y)| \leq |f_\gamma(y) - a_\gamma(y,t)| + |D^\gamma(P_{Q_{y,t}} - P_{Q_{x,2t}})(y)|
\]

\[
\leq C \int_{0}^{t} \Omega_f(y,s)s^{-|\gamma|-1} ds + Ct^{-|\gamma|}\Omega f(x,2t).
\]

Therefore,

\[
\Omega_{f_\gamma}(x,t) \leq Ct^{-n} \int_{Q_{x,2t}} dy \left( \int_{0}^{t} \Omega_f(y,s)s^{-|\gamma|-1} ds \right) + Ct^{-|\gamma|}\Omega f(x,2t)
\]

\[
\leq C \int_{0}^{t} M(\Omega_f(\cdot,s))(x)s^{-|\gamma|-1} ds + Ct^{-|\gamma|}\Omega f(x,2t),
\]

where \( M \) denotes the Hardy-Littlewood maximal operator. Now, Hardy’s inequality gives

\[
G_{\alpha-|\gamma|}(f_\gamma)(x) \leq C \left( G_{\alpha}f(x) + \left( \int_{0}^{t} (t^{-\alpha}M\Omega_f(\cdot,t))(x)^2 t^{-1} dt \right)^{1/2} \right),
\]

and the lemma follows by the Fefferman-Stein theorem on vector-valued maximal operators [5].
Lemma 2. Let \( f \in L^p \) be such that \( G_\alpha f \in L^p \), \( \alpha > 1 \). Then the weak partials \( \frac{\partial f}{\partial x_i} \) exist and coincide with \( f_{e_i} \), where \( e_i \) denotes the n-tuple with 1 in the \( i \)th place and 0 in the others.

Proof. It is enough to show that \( (f(x + te_i) - f(x))/t \) tends to \( f_{e_i}(x) \) in \( L^p \) as \( t \) goes to 0. First, we have by (8),

\[
|t^{-1}(f(x + te_i) - f(x)) - f_{e_i}(x)|
\leq t^{-1}|f(x + te_i) - a_0(x + te_i, 2t)| + t^{-1}|f(x) - a_0(x, 2t)|
+ |t^{-1}(a_0(x + te_i, 2t) - a_0(x, 2t)) - f_{e_i}(x)|
\leq Ct^{\alpha - 1}(G_\alpha f(x + te_i) + G_\alpha f(x))
+ |t^{-1}(a_0(x + te_i, 2t) - a_0(x, 2t)) - f_{e_i}(x)|.
\]

Next, since \( a_0(y, t) = P_{Q_x, t}(y) \),

\[
a_0(x + te_i, 2t) - a_0(x, 2t)
= P_{Q_{x + te_i}, 2t}(x + te_i) - P_{Q_{x, 2t}}(x + te_i) + P_{Q_{x, 2t}}(x + te_i) - P_{Q_{x, 2t}}(x)
= P_{Q_{x + te_i}, 2t}(x + te_i) - P_{Q_{x, 2t}}(x + te_i) + \sum_{j=1}^{[\alpha]} a_{je_i}(x, 2t)t^j/j!
\]

thus,

\[
|t^{-1}(a_0(x + te_i, 2t) - a_0(x, 2t)) - f_{e_i}(x)|
\leq |t^{-1}(a_0(x + te_i, 2t) - a_0(x, 2t)) - a_{e_i}(x, 2t)| + |a_{e_i}(x, 2t) - f_{e_i}(x)|
\leq t^{-1}|P_{Q_{x + te_i}, 2t}(x + te_i) - P_{Q_{x, 2t}}(x + te_i)|
+ C \sum_{j=2}^{[\alpha]} |a_{je_i}(x, 2t)|t^{j-1} + Ct^{\alpha - 1}G_\alpha f(x)
\leq Ct^{-1}t^{-n} \int_{Q_{x + te_i}, t} |P_{Q_{x + te_i}, 2t}(y) - P_{Q_{x, 2t}}(y)| dy
+ C \sum_{j=2}^{[\alpha]} t^{j-1}|a_{je_i}(x, 2t)| + Ct^{\alpha - 1}G_\alpha f(x),
\]

and taking \( L^p \) norms, we obtain

\[
\|t^{-1}(f(\cdot + te_i) - f(\cdot)) - f_{e_i}(\cdot)\|_p
\leq Ct^{-1}\|\Omega_f(\cdot, 2t)\|_p + C \sum_{j=2}^{[\alpha]} t^{j-1}\|a_{je_i}(\cdot, 2t)\|_p + Ct^{\alpha - 1}\|G_\alpha f\|_p.
\]

But \( \|\Omega_f(\cdot, 2t)\|_p \leq Ct^{\alpha}\|G_\alpha f\|_p \) and, if \( j < [\alpha], \|a_{je_i}(\cdot, 2t)\|_p \leq C(\|G_\alpha f\|_p + \|f\|_p) \), by (7), whereas if \( j = [\alpha] = \alpha \), an easy modification of (7) gives

\[
\|a_{je_i}(\cdot, 2t)\|_p \leq C(log t^{-1})(\|G_\alpha f\|_p + \|f\|_p).
\]

In any case \( \|t^{-1}(f(\cdot + te_i) - f(\cdot)) - f_{e_i}(\cdot)\|_p \) tends to 0 as \( t \) goes to 0.
To finish Theorem 5 we will prove

**Lemma 3.** Let $f$ be an $L^p$ function such that $G_\alpha f \in L^p$, $\alpha > 1$. Then $G_{\alpha - 1} f \in L^p$ and $\|G_{\alpha - 1} f\|_p \leq C(\|G_\alpha f\|_p + \|f\|_p)$.

**Proof.** Let $k = \lfloor \alpha \rfloor$. We will follow the lines of the proofs of similar facts in [3] or [6] (see also [4]). Fix $x \in \mathbb{R}^n$ and $t > 0$. For each cube $Q$ containing $x$ and such that $|Q| = t^n$, write $P_{Q}^k f$ as

$$P_{Q}^k f(y) = \sum_{|\gamma| < k} a_\gamma(Q)(y - x) + \sum_{|\gamma| = k} a_\gamma(Q)(y - x) = R_Q(y) + S_Q(y).$$

Then, by (3),

$$|Q|^{-1} \int_Q |f - P_{Q}^{k-1} f| \leq C |Q|^{-1} \int_Q |f - R_Q| \leq C \left( \Omega_f^k(x,t) + \text{ess sup}_Q |S_Q| \right).$$

Denoting by $Q_i$ the cube with same centre as $Q$ and side length $2^it$, (2) implies, for each $|\gamma| = k$,

$$|a_\gamma(Q_i)| \leq C(2^it)^{-k} |Q_i|^{-1} \int_{Q_i} |f| \leq C(2^it)^{-k-n/p} \|f\|_p,$$

that is, $a_\gamma(Q_i)$ tends to 0 as $i$ goes to $\infty$ and, again by (2),

$$|a_\gamma(Q)| \leq \sum_{i=0}^{\infty} |a_\gamma(Q_{i+1}) - a_\gamma(Q_i)| \leq C \sum_{i=0}^{\infty} (2^it)^{-k} \Omega_f^k(x,2^it).$$

Therefore, inserting this estimate in (9), we obtain

$$\Omega_f^{k-1}(x,t) \leq C \left( \Omega_f^k(x,t) + t^k \int_t^\infty \Omega_f^k(x,s)s^{-k-1} ds \right).$$

Now Hardy’s inequality gives

$$G_{\alpha - 1} f(x) \leq C \left( \int_0^\infty (t^{-\alpha+1} \Omega_f^k(x,t))^2 t^{-1} dt \right)^{1/2} \leq C \left( \left( \int_0^1 \right)^{1/2} + \left( \int_1^\infty \right)^{1/2} \right) \leq C \left( \int_0^1 (t^{-\alpha} \Omega_f^k(x,t))^2 t^{-1} dt \right)^{1/2} + CM f(x),$$

and the lemma follows.

Theorems 4 and 5 imply that, if the Theorem 2 holds for a given $\alpha$, it also holds for $\alpha + 1$. Therefore, the already proved case $0 < \alpha < 1$ implies that Theorem 2 is true for all $\alpha > 0$, $\alpha$ nonintegral, and that if we can prove it for $\alpha = 1$, it will hold for $\alpha$ integral as well. We do this in the following section.
4. The case $\alpha$ integral. Let $f$ be a locally integrable function and fix an integer $k$ and a cube $Q$ with side length $t > 0$. For any $y \in Q$ and any cube $Q'$ such that $y \in Q' \subset Q$, let $Q' = Q_0 \subset Q_1 \subset \cdots \subset Q_m = Q$ be a sequence of cubes such that $|Q_i| = 2^n|Q_{i-1}|$. By (1) and (4) we have

$$|Q'|^{-1} \int_{Q'} |f - P_{Q} f| \leq C \sum_{0}^{m} \int_{Q_i} |f - P_{Q_i} f| \leq C \int_{0}^{t} \Omega_f(y, s)s^{-1} ds$$

which by Lebesgue’s differentiation theorem gives, for a.e. $y \in Q$,

$$|f(y) - P_{Q} f(y)| = \lim_{Q' \to y} |Q'|^{-1} \int_{Q'} |f - P_{Q'} f| \leq C \int_{0}^{t} \Omega_f(y, s)s^{-1} ds. \tag{10}$$

Now, the second difference operator $\Delta^2_{Q}$ annihilates polynomials of degree 1, and for a.e. $x$ we obtain by (10)

$$t^{-n} \int_{Q_{0,t}} |\Delta^2_{Q} f(x)| dh = t^{-n} \int_{Q_{0,t}} |\Delta^2_{Q} (f - P^1_{Q_x,z_1})(x)| dh \leq C \int_{0}^{2t} M\Omega^1_f(., s)(x)s^{-1} ds,$n

which by Hardy’s inequality implies

$$S_1 f(x) = \left( \int_{0}^{\infty} \left( t^{-1} t^{-n} \int_{Q_{0,t}} |\Delta^2_{Q} f(x)| dh \right)^2 t^{-1} dt \right)^{1/2} \leq C \left( \int_{0}^{\infty} (t^{-1} M\Omega^1_f(., t)(x))^2 t^{-1} dt \right)^{1/2}.$$

Thus, if $G_1 f \in L^p$, $1 < p < \infty$, the $L^p$ boundedness of the vector-valued Hardy-Littlewood maximal operator gives that $\|S_1 f\|_p \leq C\|G_1 f\|_p$.

Conversely, to prove that $f \in L^p$ implies $G_1 f \in L^p$, we will use complex interpolation. If $(A, B)_{[\theta]}$ denotes the intermediate space between the Banach spaces $A, B$ obtained by the complex method for the index $\theta$, it is well known (see, for instance [2]) that $(L^p_{\alpha_1}, L^p_{\alpha_2})_{[\theta]} = L^p_{\alpha}$, with $\alpha = \theta \alpha_1 + (1 - \theta)\alpha_2$. Suppose now that $0 < \alpha_1 < 1 < \alpha_2 < 2$ and let $H_{\alpha_i}$, $i = 1, 2$, denote the Banach space of functions $F(y, t)$ defined in the product $(0, \infty) \times Q_{0,1}$ such that

$$\|F\|_{H_{\alpha_i}} = \left( \int_{0}^{\infty} \left( t^{-\alpha_i} \int_{Q_{0,1}} |F(y, t)| dy \right)^2 t^{-1} dt \right)^{1/2} < \infty.$$

By the results in §§2 and 3 and the inequality $\Omega^1_f \leq C\Omega_j^0$ derived from (4), the map

$$f \to F(y, t; x) = f(x + ty) - P^1_{Q_x,z_1} f(x + ty)$$

is a bounded linear map from $L^p_{\alpha_i}$ to the $L^p$ space of $H_{\alpha_i}$ valued functions $L^p(H_{\alpha_i})$ and, therefore, from $L^p_1 = (L^p_{\alpha_1}, L^p_{\alpha_2})_{[\theta]}$ into $(L^p(H_{\alpha_1}), L^p(H_{\alpha_2}))_{[\theta]}$ with $\theta =
\( \alpha_2 - 1/\alpha_2 - \alpha_1 \). But \((L^p(H_{\alpha_1}),\ L^p(H_{\alpha_2})) = L^p(H_{\alpha_1}, H_{\alpha_2}) \) and \((H_{\alpha_1}, H_{\alpha_2}) = H_1 \) \cite[pp. 107 and 122]{2}. Therefore, if \( f \in L^p \),

\[ \|G_1 f\|_p \leq C \|\|F(\cdot, x)\|_H_1\|_p \leq C \|f\|_{p, 1}. \]

This concludes the proof of Theorem 2 when \( \alpha = 1 \) and, by the preceding remarks, for all \( \alpha > 0 \).

5. Further results. First, we will show how Theorem 2 relates with the Marcinkiewicz integral characterization of \( L^p \) when \( p > 2n/n + 2\alpha \). For simplicity, we will give the details only for \( 0 < \alpha < 1 \). In this range of \( \alpha \), the Marcinkiewicz integral of a function \( f \) can be written as

\[
(D_{\alpha} f(x))^2 = \int_{\mathbb{R}^n} |f(x + y) - f(x)|^2 |y|^{-2\alpha - n} \, dy
\]

\[ = \int_{\mathbb{R}^n} \int_{|y|}^{\infty} t^{-2\alpha - n - 1} \, dt \, dy \]

\[ = \int_{0}^{\infty} t^{-2\alpha - n} \left( \int_{|y|}^{t} |f(x + y) - f(x)|^2 \, dy \right) \, dt, \]

which can be seen as a variant of \( G_{\alpha} f \). This suggests the possibility of replacing in the definition of \( G_{\alpha} \) the \( L^1 \) means \( \Omega_f(x, t) \) by the \( L^r \) means

\[
\Omega_{f, r}(x, t) = \sup \left\{ \left( |Q| - 1 \int_{Q} |f - P_Q|^r \right)^{1/r} : x \in Q, |Q| = t^n \right\}. \]

The following theorem tells us when this can be done.

**Theorem 6.** Let \( \alpha > 0 \) and \( f \) be an \( L^p \) function, \( 1 < p < \infty \), such that \( G_{\alpha} f \in L^p \). Then

(i) if \( 1 < p \leq 2 \) and \( \alpha \leq n/p \), the function \( G_{\alpha, r} f \) obtained by replacing \( \Omega_f \) with \( \Omega_{f, r} \) in the definition of \( G_{\alpha} f \) is also in \( L^p \) for \( r < pn/n - \alpha \); if \( \alpha > n/p \), \( G_{\alpha, \infty} \in L^p \);

(ii) if \( 2 \leq p < \infty \) and \( \alpha \leq n/2 \), \( G_{\alpha, r} f \in L^p \) for \( r < 2n/n - 2\alpha \); if \( \alpha > n/2 \), \( G_{\alpha, \infty} \in L^p \).

Finally, in all cases \( \|G_{\alpha, r} f\|_p \sim \|G_{\alpha} f\|_p \).

**Proof.** Fix \( x \in \mathbb{R}^n \), \( t > 0 \) and a cube \( Q \) such that \( x \in Q, |Q| = t^n \). By (10) we have a.e. in \( Q \)

\[
|f(y) - P_Q(y)| \leq C \int_{0}^{t} \Omega_f(y, s)s^{-1} \, ds \tag{11}
\]

Suppose now \( 1 < p \leq 2 \) and \( \alpha \leq n/p \), and for \( r < pn/n - \alpha \), fix \( q < p \) and \( \beta < \alpha \) such that \( 1/r = 1/q - \beta/n \). By (11),

\[
|f(y) - P_Q(y)| \leq C \int_{0}^{t} s^{-\beta} M_{\beta}(\Omega_f(\cdot, s)\chi_Q)(y)s^{-1} \, ds,
\]
where $M_{\beta}$ denotes the maximal operator $M_{\beta}g(z) = \sup\{|Q|^{-1+\beta/n} \int_Q |g| : z \in Q\}$. As it is well known, $M_{\beta}$ maps $L^q$ into $L^r$ and, therefore,

\[
\left(\int_Q |f - P_Q|^r \right)^{1/r} \leq C \int_0^t s^{-\beta} \left(\int_Q (M_{\beta}(\Omega_f(\cdot, s)\chi_Q))^r dy \right)^{1/r} s^{-1} ds
\]

which implies

\[
\Omega_{f,r}(x, t) \leq Ct^{\beta} \int_0^t s^{-\beta} M_q \Omega_f(\cdot, s)(x)s^{-1} ds,
\]

with $M_q(g) = (M(|g|q))^{1/q}$. Hardy’s inequality gives now

\[
\int_0^t (t^{-\alpha} M_q(\Omega_f(\cdot, t))(x))^{2q} dt \leq C t^{q/2} \int_0^t M_q \Omega_f(\cdot, s)(x)s^{-1} ds
\]

and the estimate $\|G_{\alpha,r}f\|_p \leq C \|G_{\alpha}f\|_p$ follows from the Fefferman-Stein theorem. The same proof works for the case $\alpha = n/2$ of (ii): taking $r < 2n/n - 2\alpha$ allows us to choose $q < 2$, and to apply the Fefferman-Stein theorem again.

If $p \leq 2$ and $\alpha > n/p$, fix $\beta$ and $q$ such that $\alpha > \beta > n/q > n/p$. From (11) we deduce for a.e. $y \in Q$

\[
|f(y) - P_Q(y)| \leq C \int_0^t s^{\beta-n/q}s^{-\beta} \left(\int_Q \Omega_f(z, s)^q dz \right)^{1/q} s^{-1} ds
\]

(12)

\[
\leq Ct^{\beta-n/q} \int_0^t s^{-\beta} \left(\int_Q \Omega_f(z, s)^q dz \right)^{1/q} s^{-1} ds
\]

\[
\leq Ct^{\beta} \int_0^t s^{-\beta} M_q \Omega_f(\cdot, s)(x)s^{-1} ds
\]

and the inequality $\|G_{\alpha,\infty}f\|_p \leq C \|G_{\alpha}f\|_p$ follows as before. The same argument works for the case $\alpha > n/2$ of (ii). Since the estimate $\|G_{\alpha,r}f\|_p \leq C \|G_{\alpha,r}f\|_p$ is obvious, Theorem 6 is proved.

As a consequence, $G_{\alpha,2}f$ characterizes $L^p_\alpha$ for $p > 2n/n + 2\alpha$ and, since $D_{\alpha}f \sim G_{\alpha,2}f$, so does $D_{\alpha}f$.

Finally, the characterization of potential spaces in terms of the function $G_{\alpha}$ provides easy proofs of the well-known embeddings $L^p_\alpha \subset L^q_B$, $1/p - \alpha/n = 1/q - \beta/n$, and can be used to show that, for $\alpha > n/p$, $L^p_\alpha$ is an algebra under multiplication. Also, Besov spaces $B^{p,q}_\alpha$ of $L^p$ functions (see [8] for the definitions) can be characterized by the finiteness of the expression

\[
\left(\int_0^\infty (t^{-\alpha}\Omega_f^{[\alpha]}(\cdot, t))q t^{-1} dt \right)^{1/q}.
\]

Combining this with Theorem 2, easy proofs of the embeddings $B^{p,p}_\alpha \subset L^p_\alpha \subset B^{p,2}_\alpha$, $1 < p \leq 2$, $B^{p,2}_\alpha \subset L^p_\alpha \subset B^{p,p}_\alpha$, $2 \leq p < \infty$ and $B^{p,\infty}_\alpha \subset L^p_{\alpha-\epsilon}$ for all $\epsilon > 0$ follow.
REFERENCES


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