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A CHARACTERIZATION OF POTENTIAL SPACES

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ABSTRACT. A mean oscillation characterization, valid for all $\alpha > 0$, of the spaces L^p_α of Bessel potentials of L^p functions is given and is used to relate the known characterizations for $0 < \alpha < 2$ via Marcinkiewicz integrals, due to E. M. Stein, and via vector-valued means of differences, due to R. S. Strichartz.

1. Introduction. The Bessel potential of order α, $\alpha > 0$, of a smooth function g is defined as $J_\alpha g = K_\alpha * g$, where

$$(K_\alpha)\hat{}(\xi) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}.$$

The kernel K_α can be shown to be integrable and, for $1 \leq p \leq \infty$, the potential spaces L^p_α, introduced by Aronszajn and Smith and by Calderón, are then the spaces of Bessel potentials of order α of L^p functions: that is,

$$L^p_\alpha = \{J_\alpha g \colon g \in L^p\}.$$

With the norm $\|J_\alpha g\|_{p,\alpha} = \|g\|_p$, L^p_α becomes a Banach space.

This paper is concerned with the question of when a given function f is the Bessel potential of an L^p function. An obviously necessary condition is that f be in L^p, and Stein proved [**7**], for $0 < \alpha < 2$ and $2n/n + 2\alpha < p < \infty$, that an L^p function f belongs to L^p_α if and only if its Marcinkiewicz integral

$$D_\alpha f(x) = \left(\int_{\mathbf{R}^n} |\Delta_y^{[\alpha]+1} f(x)|^2 \, |y|^{-2\alpha-n} \, dy\right)^{1/2}$$

is in L^p and, furthermore, $\|f\|_{p,\alpha} \sim \|f\|_p + \|D_\alpha f\|_p$ (by $A \sim B$ we mean there is a constant C independent of A or B such that $C^{-1}A \leq B \leq CA$; also, $\Delta_y f(x) = f(x+y) - f(x)$).

With p now such that $1 < p < \infty$, the following characterization of Bessel potentials for the same range of α is due to Strichartz [**9**]. Let $[\alpha]$ denote the integral part of α.

THEOREM 1. *Let $0 < \alpha < 2$ and $1 < p < \infty$. An L^p function f belongs to L^p_α if and only if the function*

$$S_\alpha f(x) = \left(\int_0^\infty \left(t^{-\alpha} \int_{|y| \leq 1} |\Delta_{ty}^{[\alpha]+1} f(x)| \, dy\right)^2 t^{-1} \, dt\right)^{1/2}$$

belongs to L^p. Furthermore, $\|f\|_{p,\alpha} \sim \|f\|_p + \|S_\alpha f\|_p$.

This result was extended by Bagby [**1**] to the range $0 < \alpha < n$.

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We will show here how Bessel potentials of L^p functions, $1 < p < \infty$, can be characterized for a general $\alpha > 0$. This will be done by means of a mixed norm estimate of the approximation of these functions by polynomials. To be more precise, let f be a locally integrable function and Q a cube in $\mathbf{R}^n$. We denote by $P_Q^k f$ the unique polynomial of degree k such that

$$\int_Q (f(y) - P_Q^k f(y)) y^\gamma \, dy = 0$$

for each n-tuple $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbf{N}^n$ such that $|\gamma| = \gamma_1 + \cdots + \gamma_n \leq k$ (for instance, if $k = 0$, $P_Q^0 f$ is just the mean $f_Q = |Q|^{-1} \int_Q f$). We will write $P_Q f$ or even P_Q if there is no chance of confusion. Then, for $x \in \mathbf{R}^n$ and $t > 0$, we define

$$\Omega_f^k(x,t) = \Omega_f(x,t) = \sup \left\{ |Q|^{-1} \int_Q |f - P_Q^k f| \colon x \in Q, |Q| = t^n \right\}.$$

The main result of this paper is

THEOREM 2. *Let $\alpha > 0$ and $f \in L^p$, $1 < p < \infty$. Then $f \in L_\alpha^p$ if and only if the function*

$$G_\alpha f(x) = \left(\int_0^\infty (t^{-\alpha} \Omega_f^{[\alpha]}(x,t))^2 t^{-1} \, dt \right)^{1/2}$$

is in L^p. Furthermore, $\|f\|_{p,\alpha} \sim \|f\|_p + \|G_\alpha f\|_p$.

The paper is organized as follows: §2 contains the proof of Theorem 2 when $0 < \alpha < 1$; in view of Strichartz's theorem, we will prove that $G_\alpha f \in L^p$ if and only if $S_\alpha f \in L^p$. In §3 we deal with the case $\alpha > 0$, α nonintegral, by reducing it to that of §2. Complex interpolation is then used in §4 to deal with the remaining case $\alpha > 0$, α integral. Finally, §5 contains the relationship between $G_\alpha f$ and the Marcinkiewicz integral $D_\alpha f$, and some further results.

2. The case $0 < \alpha < 1$. We begin with some remarks about the polynomials P_Q. Let f be a locally integrable function. An easy homogeneity argument gives [**6**]

$$\operatorname*{ess\,sup}_Q |P_Q^k f| \leq C|Q|^{-1} \int_Q |f|, \tag{1}$$

for any cube Q and $k \geq 0$, and where C does not depend on Q or f (throughout the paper C will denote any absolute constant independent of the particular functions, points or sets considered). As a consequence, for any n-tuple $\gamma \in \mathbf{N}^n$, we have for the γ-derivatives of P_Q (see [**3**, **4**])

$$\begin{aligned} \operatorname*{ess\,sup}_Q |D^\gamma (P_Q^k f)| &\leq C|Q|^{-\gamma/n} \operatorname*{ess\,sup}_Q |P_Q^k f| \\ &\leq C|Q|^{-\gamma/n-1} \int_Q |f|. \end{aligned} \tag{2}$$

It also follows from (1) that, since $P_Q^k(f + R) = P_Q^k f + R$ for any polynomial R of degree $\leq k$,

$$|Q|^{-1} \int_Q |f - P_q^k f| \leq C|Q|^{-1} \int_Q |f - R|, \tag{3}$$

and that if $Q \subset Q'$,

$$|Q|^{-1} \int_Q |f - P_Q^k f| \le C(|Q'|/|Q|)|Q'|^{-1} \int_{Q'} |f - P_{Q'}^k f|. \tag{4}$$

The cube with centre x and side length t will be denoted by $Q_{x,t}$. From (4) we obtain

$$\Omega_f^k(x,t) \le Ct^{-n} \int_{Q_{x,2t}} |f - P_{Q_{x,2t}}|, \tag{5}$$

and therefore, if $t \le s \le 2t$, $\Omega_f^k(x,t) \le C\Omega_f^k(x,s) \le C\Omega_f^k(x,2t)$. In particular, the series

$$\left(\sum_{-\infty}^{\infty} (2^{-i\alpha}\Omega_f(x,2^i))^2\right)^{1/2}$$

is comparable to G_α, and we can also use balls instead of cubes to define Ω_f and $G_\alpha f$. We have now

THEOREM 3. *Let $\alpha > 0$, α nonintegral, and $f \in L^p$, $1 < p < \infty$. Then $G_\alpha f \in L^p$ if and only if for a.e. $x \in \mathbf{R}^n$ there is a polynomial P_x of degree $\le [\alpha]$, depending on f, such that if*

$$\tilde{G}_\alpha f(x) = \left(\int_0^\infty \left(t^{-\alpha-n} \int_{Q_{x,t}} |f(y) - P_x(y)|\, dy\right)^2 t^{-1}\, dt\right)^{1/2},$$

$\tilde{G}_\alpha f \in L^p$. Furthermore, $\|G_\alpha f\|_p \sim \|\tilde{G}_\alpha f\|_p$. The polynomial P_x will be a.e. the Taylor polynomial of f.

PROOF. The inequality $G_\alpha f \le C\tilde{G}_\alpha f$ is an immediate consequence of (5) and (3). Conversely, if $G_\alpha f \in L^p$, fix $x \in \mathbf{R}^n$, $\gamma \in \mathbf{N}^n$ with $|\gamma| \le [\alpha]$ and write $P_{Q_{x,t}}^{[\alpha]} f(y) = P_{Q_{x,t}}(y)$ as

$$P_{Q_{x,t}}(y) = \sum_{|\beta| \le [\alpha]} a_\beta(x,t)(y-x)^\beta/\beta!.$$

If $t > s > 0$, let $Q_0 \subset Q_1 \subset \cdots \subset Q_m$ be a sequence of cubes such that $Q_0 = Q_{x,s}$, $Q_m = Q_{x,t}$ and $|Q_i| = 2^n|Q_{i-1}|$. Then, by (2),

$$\begin{aligned} |a_\gamma(x,t) - a_\gamma(x,s)| &= |D^\gamma(P_{Q_0} - P_{Q_m})(x)| \\ &\le \sum_1^m |D^\gamma(P_{Q_i} - P_{Q_{i-1}})(x)| \\ &\le C\sum_1^m |Q_i|^{-1-|\gamma|/n} \int_{Q_i} |f - P_{Q_i}| \\ &\le C\int_s^t \Omega_f(x,u)u^{-|\gamma|-1}\, du. \end{aligned}$$

Thus, if $\alpha > |\gamma|$, Schwartz's inequality gives

$$|a_\gamma(x,t) - a_\gamma(x,s)| \le Ct^{\alpha-|\gamma|}G_\alpha f(x). \tag{6}$$

In addition, we have

$$\begin{aligned}|a_\gamma(x,t)| &\le |a_\gamma(x,t)-a_\gamma(x,1)|+|a_\gamma(x,1)| \\ &\le C(G_\alpha f(x)+Mf(x)),\end{aligned} \tag{7}$$

with M the Hardy-Littlewood maximal operator. Therefore, as t goes to 0, the functions $a_\gamma(x,t)$ form a Cauchy sequence of L^p functions. We denote its L^p limit by f_γ, which will also be a.e. the pointwise limit of some subsequence $a_\gamma(x,t_j)$. Hence, for a.e. x

$$\begin{aligned}|f_\gamma(x)-a_\gamma(x,t)| &= \lim|a_\gamma(x,t_j)-a_\gamma(x,t)| \\ &\le C\int_0^t \Omega_f(x,u)u^{-|\gamma|-1}\,du \le Ct^{\alpha-|\gamma|}G_\alpha f(x),\end{aligned} \tag{8}$$

and, in particular, $\|f_\gamma\|_p \le \|f_\gamma - a_\gamma(\cdot,1)\|_p + \|a_\gamma(\cdot,1)\|_p \le C(\|G_\alpha f\|_p + \|f\|_p)$.

If we now define $P_x(y)=\sum_{|\gamma|\le k} f_\gamma(x)(y-x)^\gamma/\gamma!$, we obtain

$$\begin{aligned}\int_{Q_{x,t}}|f-P_x|\,dy &\le \int_{Q_{x,t}}|f-P_{Q_{x,t}}|\,dy + C\sum_{|\gamma|\le k}|a_\gamma(x,t)-f_\gamma(x)|t^{|\gamma|} \\ &\le Ct^{-n}\left(\Omega_f(x,t)+\sum_{i=0}^k t^i\int_0^t \Omega_f(x,u)u^{-i-1}\,du\right).\end{aligned}$$

Assuming that $k=[\alpha]<\alpha$, Hardy's inequality implies

$$\tilde{G}_\alpha f(x) \le CG_\alpha f(x) \quad \text{for a.e. } x\in\mathbf{R}^n.$$

In particular, since $a_0(x,t)=f_{Q_{x,t}}$ tends to $f(x)$ a.e., it follows that for $0<\alpha<1$, $\tilde{G}_\alpha f(x)=S_\alpha f(x)$ a.e., and Theorem 2 is proved in this case.

3. The case $\alpha>0$, α nonintegral. The following is a well-known result in the theory of potential spaces (see, for instance [**8**, p. 136]).

THEOREM 4. *Let $f\in L^p$, $1<p<\infty$. Then $f\in L^p_\alpha$, $\alpha\ge 1$, if and only if f and its weak partials $\partial f/\partial x_i$ belong to $L^p_{\alpha-1}$. Furthermore, $\|f\|_{p,\alpha}\sim\|f\|_{p,\alpha-1}+\sum_1^n\|\partial f/\partial x_i\|_{p,\alpha-1}$.*

A similar result also holds for L^p functions f such that $G_\alpha f\in L^p$; concretely,

THEOREM 5. *Let $f\in L^p$, $1<p<\infty$. Then, for $\alpha>1$, $G_\alpha f\in L^p$ if and only if $G_{\alpha-1}f\in L^p$, the weak partials $\partial f/\partial x_i\in L^p$ and $G_{\alpha-1}(\partial f/\partial x_i)\in L^p$. Furthermore,*

$$\|f\|_p+\|G_\alpha f\|_p \sim \|f\|_p+\|G_{\alpha-1}f\|_p+\sum_1^n(\|\partial f/\partial x_i\|_p+\|G_{\alpha-1}(\partial f/\partial x_i)\|_p).$$

PROOF. We begin with the only if part. Suppose first that f is a C^1 function such that f and $\partial f/\partial x_i$ are in L^p, and fix x and $t>0$. For each $i=1,\dots,n$ and $y\in Q_{0,t}$, consider the polynomials

$$P_i(x+y)=\int_0^1 P^{k-1}_{Q_{x,st}}(\partial f/\partial x_i)(x+sy)\,ds,$$

with $k = [\alpha]$. Then, by Taylor's formula,

$$t^{-n}\int_{Q_{x,t}}\left|f(z)-f(x)-\sum_1^n P_i(z)(z_i-x_i)\right|dz$$
$$\leq t^{-n}\sum_1^n\int_0^1 t\int_{Q_{0,t}}|\partial f/\partial x_i(x+sy)-P^{k-1}_{Q_{x,st}}(\partial f/\partial x_i)(x+sy)|\,dy\,ds$$
$$\leq C\sum_1^n t\int_0^1 (st)^{-n}\int_{Q_{0,st}}|\partial f/\partial x_i(x+z)-P^{k-1}_{Q_{x,st}}(x+z)|\,dz\,ds,$$

which implies

$$\Omega_f^k(x,t)\leq C\sum_1^n\int_0^{2t}\Omega^{k-1}_{\partial f/\partial x_i}(x,u)\,du.$$

This estimate can be extended to any $f\in L^p$ with weak partials in L^p by a standard regularization argument and, in the conditions of the theorem, Hardy's inequality yields

$$G_\alpha f(x)\leq C\sum_1^n G_{\alpha-1}(\partial f/\partial x_i)(x).$$

Next, we will divide the proof of the if part into three lemmas.

LEMMA 1. *Let $f\in L^p$ such that $G_\alpha f\in L^p$, and $\gamma\in\mathbf{N}^n$, $|\gamma|<\alpha$. Then, with f_γ as in Theorem 3, $\|G_{\alpha-|\gamma|}f_\gamma\|_p\leq C\|G_\alpha f\|_p$.*

PROOF. Fix $x\in\mathbf{R}^n$ and a cube Q with $x\in Q$ and $|Q|=t^n$. If $y\in Q$, $Q_{y,t}\subset Q_{x,2t}$, and by (8) and (2),

$$|f_\gamma(y)-D^\gamma P_{Q_{x,2t}}(y)|\leq|f_\gamma(y)-a_\gamma(y,t)|+|D^\gamma(P_{Q_{y,t}}-P_{Q_{x,2t}})(y)|$$
$$\leq C\int_0^t\Omega_f(y,s)s^{-|\gamma|-1}\,ds+Ct^{-|\gamma|}\Omega f(x,2t).$$

Therefore,

$$\Omega_{f_\gamma}(x,t)\leq Ct^{-n}\int_{Q_{x,2t}}dy\left(\int_0^t\Omega_f(y,s)s^{-|\gamma|-1}\,ds\right)+Ct^{-|\gamma|}\Omega_f(x,2t)$$
$$\leq C\int_0^t M(\Omega_f(\cdot,s))(x)s^{-|\gamma|-1}\,ds+Ct^{-|\gamma|}\Omega_f(x,2t),$$

where M denotes the Hardy-Littlewood maximal operator. Now, Hardy's inequality gives

$$G_{\alpha-|\gamma|}(f_\gamma)(x)\leq C\left(G_\alpha f(x)+\left(\int_0^\infty(t^{-\alpha}M\Omega_f(\cdot,t))(x)^2t^{-1}\,dt\right)^{1/2}\right),$$

and the lemma follows by the Fefferman-Stein theorem on vector-valued maximal operators [**5**].

LEMMA 2. *Let $f \in L^p$ be such that $G_\alpha f \in L^p$, $\alpha > 1$. Then the weak partials $\partial f/\partial x_i$ exist and coincide with f_{e_i}, where e_i denotes the n-tuple with 1 in the ith place and 0 in the others.*

PROOF. It is enough to show that $(f(x+te_i)-f(x))/t$ tends to $f_{e_i}(x)$ in L^p as t goes to 0. First, we have by (8),

$$\begin{aligned}
&|t^{-1}(f(x+te_i)-f(x))-f_{e_i}(x)| \\
&\quad \le t^{-1}|f(x+te_i)-a_0(x+te_i,2t)|+t^{-1}|f(x)-a_0(x,2t)| \\
&\qquad + |t^{-1}(a_0(x+te_i,2t)-a_0(x,2t))-f_{e_i}(x)| \\
&\quad \le Ct^{\alpha-1}(G_\alpha f(x+te_i)+G_\alpha f(x)) \\
&\qquad + |t^{-1}(a_0(x+te_i,2t)-a_0(x,2t))-f_{e_i}(x)|.
\end{aligned}$$

Next, since $a_0(y,t)=P_{Q_{y,t}}(y)$,

$$\begin{aligned}
&a_0(x+te_i,2t)-a_0(x,2t) \\
&\quad = P_{Q_{x+te_i,2t}}(x+te_i)-P_{Q_{x,2t}}(x+te_i)+P_{Q_{x,2t}}(x+te_i)-P_{Q_{x,2t}}(x) \\
&\quad = P_{Q_{x+2te_i,2t}}(x+te_i)-P_{Q_{x,2t}}(x+te_i)+\sum_{j=1}^{[\alpha]} a_{je_i}(x,2t)t^j/j!;
\end{aligned}$$

thus,

$$\begin{aligned}
&|t^{-1}(a_0(x+te_i,2t)-a_0(x,2t))-f_{e_i}(x)| \\
&\quad \le |t^{-1}(a_0(x+te_i,2t)-a_0(x,2t))-a_{e_i}(x,2t)|+|a_{e_i}(x,2t)-f_{e_i}(x)| \\
&\quad \le t^{-1}|P_{Q_{x+te_i,2t}}(x+te_i)-P_{Q_{x,2t}}(x+te_i)| \\
&\qquad + C\sum_{j=2}^{[\alpha]} |a_{je_i}(x,2t)|t^{j-1}+Ct^{\alpha-1}G_\alpha f(x) \\
&\quad \le Ct^{-1}t^{-n}\int_{Q_{x+te_i,t}} |PQ_{x+te_i,2t}(y)-P_{Q_{x,2t}}(y)|\,dy \\
&\qquad + C\sum_{j=2}^{[\alpha]} t^{j-1}|a_{je_i}(x,2t)|+Ct^{\alpha-1}G_\alpha f(x),
\end{aligned}$$

and taking L^p norms, we obtain

$$\begin{aligned}
&\|t^{-1}(f(\cdot+te_i)-f(\cdot))-f_{e_i}(\cdot)\|_p \\
&\quad \le Ct^{-1}\|\Omega_f(\cdot,2t)\|_p+C\sum_{j=2}^{[\alpha]} t^{j-1}\|a_{je_i}(\cdot,2t)\|_p+Ct^{\alpha-1}\|G_\alpha f\|_p.
\end{aligned}$$

But $\|\Omega_f(\cdot,2t)\|_p \le Ct^\alpha\|G_\alpha f\|_p$ and, if $j<[\alpha]$, $\|a_{je_i}(\cdot,2t)\|_p \le C(\|G_\alpha f\|_p+\|f\|_p)$, by (7), whereas if $j=[\alpha]=\alpha$, an easy modification of (7) gives

$$\|a_{je_i}(\cdot,2t)\|_p \le C(\log t^{-1})(\|G_\alpha f\|_p+\|f\|_p).$$

In any case $\|t^{-1}(f(\cdot,+te_i)-f(\cdot))-f_{e_i}(\cdot)\|_p$ tends to 0 as t goes to 0.

To finish Theorem 5 we will prove

LEMMA 3. *Let f be an L^p function such that $G_\alpha f \in L^p$, $\alpha > 1$. Then $G_{\alpha-1} f \in L^p$ and $\|G_{\alpha-1} f\|_p \le C(\|G_\alpha f\|_p + \|f\|_p)$.*

PROOF. Let $k = [\alpha]$. We will follow the lines of the proofs of similar facts in [**3**] or [**6**] (see also [**4**]). Fix $x \in \mathbf{R}^n$ and $t > 0$. For each cube Q containing x and such that $|Q| = t^n$, write $P_Q^k f$ as

$$P_Q^k f(y) = \sum_{|\gamma|<k} a_\gamma(Q)(y-x)^\gamma + \sum_{|\gamma|=k} a_\gamma(Q)(y-x)^\gamma = R_Q(y) + S_Q(y).$$

Then, by (3),

$$\begin{aligned} |Q|^{-1}\int_Q |f - P_Q^{k-1} f| &\le C|Q|^{-1}\int_Q |f - R_Q| \\ &\le C\left(\Omega_f^k(x,t) + \operatorname*{ess\,sup}_Q |S_Q|\right). \end{aligned} \tag{9}$$

Denoting by Q_i the cube with same centre as Q and side length $2^i t$, (2) implies, for each $|\gamma| = k$,

$$|a_\gamma(Q_i)| \le C(2^i t)^{-k}|Q_i|^{-1}\int_{Q_i} |f| \le C(2^i t)^{-k-n/p}\|f\|_p;$$

that is, $a_\gamma(Q_i)$ tends to 0 as i goes to ∞ and, again by (2),

$$|a_\gamma(Q)| \le \sum_{i=0}^{\infty} |a_\gamma(Q_{i+1}) - a_\gamma(Q_i)| \le C\sum_0^\infty (2^i t)^{-k}\Omega_f^k(x, 2^i t).$$

Therefore, inserting this estimate in (9), we obtain

$$\Omega_f^{k-1}(x,t) \le C\left(\Omega_f^k(x,t) + t^k\int_t^\infty \Omega_f^k(x,s)s^{-k-1}\,ds\right).$$

Now Hardy's inequality gives

$$\begin{aligned} G_{\alpha-1} f(x) &\le C\left(\int_0^\infty (t^{-\alpha+1}\Omega_f^k(x,t))^2 t^{-1}\,dt\right)^{1/2} \\ &\le C\left(\left(\int_0^1\right)^{1/2}\right) + C\left(\left(\int_1^\infty\right)^{1/2}\right) \\ &\le C\left(\int_0^1 (t^{-\alpha}\Omega_f^k(x,t))^2 t^{-1}\,dt\right)^{1/2} + CMf(x), \end{aligned}$$

and the lemma follows.

Theorems 4 and 5 imply that, if the Theorem 2 holds for a given α, it also holds for $\alpha + 1$. Therefore, the already proved case $0 < \alpha < 1$ implies that Theorem 2 is true for all $\alpha > 0$, α nonintegral, and that if we can prove it for $\alpha = 1$, it will hold for α integral as well. We do this in the following section.

4. The case α integral. Let f be a locally integrable function and fix an integer k and a cube Q with side length $t > 0$. For any $y \in Q$ and any cube Q' such that $y \in Q' \subset Q$, let $Q' = Q_0 \subset Q_1 \subset \cdots \subset Q_m = Q$ be a sequence of cubes such that $|Q_i| = 2^n |Q_{i-1}|$. By (1) and (4) we have

$$|Q'|^{-1} \int_{Q'} |f - P_Q^k f| \leq C \sum_0^m |Q_i|^{-1} \int_{Q_i} |f - P_{Q_i}| \leq C \int_0^t \Omega_f(y,s) s^{-1}\, ds$$

which by Lebesgue's differentiation theorem gives, for a.e. $y \in Q$,

$$\begin{aligned} |f(y) - P_Q(y)| &= \lim_{Q' \to y} |Q'|^{-1} \int_{Q'} |f - P_Q| \\ &\leq C \int_0^t \Omega_f(y,s) s^{-1}\, ds. \end{aligned} \tag{10}$$

Now, the second difference operator Δ_h^2 annihilates polynomials of degree 1, and for a.e. x we obtain by (10)

$$\begin{aligned} t^{-n} \int_{Q_{0,t}} |\Delta_h^2 f(x)|\, dh &= t^{-n} \int_{Q_{0,t}} |\Delta_h^2 (f - P^1_{Q_{x,2t}})(x)|\, dh \\ &\leq C \int_0^{2t} M\Omega_f^1(\cdot, s)(x) s^{-1}\, ds, \end{aligned}$$

which by Hardy's inequality implies

$$\begin{aligned} S_1 f(x) &= \left(\int_0^\infty \left(t^{-1} t^{-n} \int_{Q_{0,t}} |\Delta_h^2 f(x)|\, dh \right)^2 t^{-1}\, dt \right)^{1/2} \\ &\leq C \left(\int_0^\infty (t^{-1} M\Omega_f^1(\cdot, t)(x))^2 t^{-1}\, dt \right)^{1/2}. \end{aligned}$$

Thus, if $G_1 f \in L^p$, $1 < p < \infty$, the L^p boundedness of the vector-valued Hardy-Littlewood maximal operator gives that $\|S_1 f\|_p \leq C \|G_1 f\|_p$.

Conversely, to prove that $f \in L_1^p$ implies $G_1 f \in L^p$, we will use complex interpolation. If $(A, B)_{[\theta]}$ denotes the intermediate space between the Banach spaces A, B obtained by the complex method for the index θ, it is well known (see, for instance [**2**]) that $(L^p_{\alpha_1}, L^p_{\alpha_2})_{[\theta]} = L^p_\alpha$, with $\alpha = \theta\alpha_1 + (1-\theta)\alpha_2$. Suppose now that $0 < \alpha_1 < 1 < \alpha_2 < 2$ and let H_{α_i}, $i = 1, 2$, denote the Banach space of functions $F(y,t)$ defined in the product $(0,\infty) \times Q_{0,1}$ such that

$$\|F\|_{H_{\alpha_i}} = \left(\int_0^\infty \left(t^{-\alpha_i} \int_{Q_{0,1}} |F(y,t)|\, dy \right)^2 t^{-1}\, dt \right)^{1/2} < \infty.$$

By the results in §§2 and 3 and the inequality $\Omega_f^1 \leq C\Omega_f^0$ derived from (4), the map

$$f \to F(y,t;x) = f(x+ty) - P^1_{Q_{x,t}} f(x+ty)$$

is a bounded linear map from $L^p_{\alpha_i}$ to the L^p space of H_{α_i} valued functions $L^p(H_{\alpha_i})$ and, therefore, from $L_1^p = (L^p_{\alpha_1}, L^p_{\alpha_2})_{[\theta]}$ into $(L^p(H_{\alpha_1}), L^p(H_{\alpha_2}))_{[\theta]}$ with $\theta =$

$\alpha_2 - 1/\alpha_2 - \alpha_1$. But $(L^p(H_{\alpha_1}), L^p(H_{\alpha_2}))_{[\theta]} = L^p((H_{\alpha_1}, H_{\alpha_2})_{[\theta]})$ and $(H_{\alpha_1}, H_{\alpha_2})_{[\theta]} = H_1$ [**2**, pp. 107 and 122]. Therefore, if $f \in L^p_1$,

$$\|G_1 f\|_p \le C\|(\|F(\cdot,\cdot;x)\|_{H_1})\|_p \le C\|f\|_{p,1}.$$

This concludes the proof of Theorem 2 when $\alpha = 1$ and, by the preceding remarks, for all $\alpha > 0$.

5. Further results. First, we will show how Theorem 2 relates with the Marcinkiewicz integral characterization of L^p_α when $p > 2n/n + 2\alpha$. For simplicity, we will give the details only for $0 < \alpha < 1$. In this range of α, the Marcinkiewicz integral of a function f can be written as

$$\begin{aligned}(D_\alpha f(x))^2 &= \int_{\mathbf{R}^n} |f(x+y) - f(x)|^2\, |y|^{-2\alpha - n}\, dy \\ &= \int_{\mathbf{R}^n} |f(x+y) - f(x)|^2 \int_{|y|}^\infty t^{-2\alpha-n-1}\, dt\, dy \\ &= \int_0^\infty t^{-2\alpha - n}\left(\int_{|y|\le t} |f(x+y) - f(x)|^2\, dy\right) t^{-1}\, dt,\end{aligned}$$

which can be seen as a variant of $G_\alpha f$. This suggests the possibility of replacing in the definition of G_α the L^1 means $\Omega_f(x,t)$ by the L^r means

$$\Omega_{f,r}(x,t) = \sup\left\{\left(|Q|^{-1}\int_Q |f - P_Q|^r\right)^{1/r} : x \in Q, |Q| = t^n\right\}.$$

The following theorem tells us when this can be done.

THEOREM 6. *Let $\alpha > 0$ and f be an L^p function, $1 < p < \infty$, such that $G_\alpha f \in L^p$. Then*

(i) *if $1 < p \le 2$ and $\alpha \le n/p$, the function $G_{\alpha,r} f$ obtained by replacing Ω_f with $\Omega_{f,r}$ in the definition of $G_\alpha f$ is also in L^p for $r < pn/n - p\alpha$; if $\alpha > n/p$, $G_{\alpha,\infty} \in L^p$;*

(ii) *if $2 \le p < \infty$ and $\alpha \le n/2$, $G_{\alpha,r} f \in L^p$ for $r < 2n/n - 2\alpha$; if $\alpha > n/2$, $G_{\alpha,\infty} f \in L^p$.*

Finally, in all cases $\|G_{\alpha,r} f\|_p \sim \|G_\alpha f\|_p$.

PROOF. Fix $x \in \mathbf{R}^n$, $t > 0$ and a cube Q such that $x \in Q$, $|Q| = t^n$. By (10) we have a.e. in Q

$$\begin{aligned}|f(y) - P_Q(y)| &\le C\int_0^t \Omega_f(y,s)s^{-1}\, ds \\ &\le C\int_0^t \left(s^{-n}\int_{Q_{y,s}} \Omega_f(z,s)\, dz\right) s^{-1}\, ds.\end{aligned} \tag{11}$$

Suppose now $1 < p \le 2$ and $\alpha \le n/p$, and for $r < pn/n - p\alpha$, fix $q < p$ and $\beta < \alpha$ such that $1/r = 1/q - \beta/n$. By (11),

$$|f(y) - P_Q(y)| \le C\int_0^t s^{-\beta} M_\beta(\Omega_f(\cdot,s)\chi_Q)(y)s^{-1}\, ds,$$

where M_β denotes the maximal operator $M_\beta g(z) = \sup\{|Q|^{-1+\beta/n}\int_Q |g| \colon z \in Q\}$. As it is well known, M_β maps L^q into L^r and, therefore,

$$\left(\int_Q |f - P_Q|^r\right)^{1/r} \le C\int_0^t s^{-\beta}\left(\int_Q (M_\beta(\Omega_f(\cdot,s)\chi_Q))^r\,dy\right)^{1/r} s^{-1}\,ds$$
$$\le C\int_0^t s^{-\beta}\left(\int_Q \Omega_f(z,s)^q\,dz\right)^{1/q} s^{-1}\,ds$$

which implies

$$\Omega_{f,r}(x,t) \le Ct^\beta \int_0^t s^{-\beta} M_q\Omega_f(\cdot,s)(x)s^{-1}\,ds,$$

with $M_q(g) = (M(|g|^q))^{1/q}$. Hardy's inequality gives now

$$G_{\alpha,r}f(x) \le C\left(\int_0^\infty (t^{-\alpha}M_q(\Omega_f(\cdot,t))(x))^2 t^{-1}\,dt\right)^{1/2}$$

and the estimate $\|G_{\alpha,r}f\|_p \le C\|G_\alpha f\|_p$ follows from the Fefferman-Stein theorem. The same proof works for the case $\alpha \le n/2$ of (ii): taking $r < 2n/n - 2\alpha$ allows us to choose $q < 2$, and to apply the Fefferman-Stein theorem again.

If $p \le 2$ and $\alpha > n/p$, fix β and q such that $\alpha > \beta > n/q > n/p$. From (11) we deduce for a.e. $y \in Q$

$$\begin{aligned} |f(y) - P_Q(y)| &\le C\int_0^t s^{\beta - n/q} s^{-\beta}\left(\int_{Q_{y,s}} \Omega_f(z,s)^q\,dt\right)^{1/q} s^{-1}\,ds \\ &\le Ct^{\beta-n/q}\int_0^t s^{-\beta}\left(\int_Q \Omega_f(z,s)^q\,dz\right)^{1/q} s^{-1}\,ds \\ &\le Ct^\beta \int_0^t s^{-\beta}M_q\Omega_f(\cdot,s)(x)s^{-1}\,ds \end{aligned} \tag{12}$$

and the inequality $\|G_{\alpha,\infty}f\|_p \le C\|G_\alpha f\|_p$ follows as before. The same argument works for the case $\alpha > n/2$ of (ii). Since the estimate $\|G_\alpha f\|_p \le C\|G_{\alpha,r}f\|_p$ is obvious, Theorem 6 is proved.

As a consequence, $G_{\alpha,2}f$ characterizes L^p_α for $p > 2n/n + 2\alpha$ and, since $D_\alpha f \sim G_{\alpha,2}f$, so does $D_\alpha f$.

Finally, the characterization of potential spaces in terms of the function G_α provides easy proofs of the well-known imbeddings $L^p_\alpha \subset L^q_\beta$, $1/p - \alpha/n = 1/q - \beta/n$, and can be used to show that, for $\alpha > n/p$, L^p_α is an algebra under multiplication. Also, Besov spaces $B^{p,q}_\alpha$ of L^p functions (see [8] for the definitions) can be characterized by the finiteness of the expression

$$\left(\int_0^\infty (t^{-\alpha}\|\Omega_f^{[\alpha]}(\cdot,t)\|_p)^q t^{-1}\,dt\right)^{1/q}.$$

Combining this with Theorem 2, easy proofs of the imbeddings $B^{p,p}_\alpha \subset L^p_\alpha \subset B^{p,2}_\alpha$, $1 < p \le 2$, $B^{p,2}_\alpha \subset L^p_\alpha \subset B^{p,p}_\alpha$, $2 \le p < \infty$ and $B^{p,\infty}_\alpha \subset L^p_{\alpha-\varepsilon}$ for all $\varepsilon > 0$ follow.

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