

ROTATION INVARIANT IDEALS IN SUBALGEBRAS OF L^∞

PAMELA GORKIN¹

ABSTRACT. In this paper, it is shown that the only (nontrivial) finitely generated rotation invariant ideals in H^∞ are $z^n H^\infty$ for some positive integer n . Using results about function algebras, it is shown that other rotation invariant ideals exist. Rotation invariant ideals of other subalgebras of L^∞ are also studied.

Let D and ∂D be the open unit disc and its boundary, respectively, in the complex plane C . Let L^∞ be the Banach algebra of all essentially bounded Lebesgue measurable functions on ∂D with the essential supremum norm. If f is a function in L^∞ and λ is a point in ∂D , let f_λ denote the L^∞ function defined by $f_\lambda(z) = f(\lambda z)$. Let B denote a closed subalgebra of L^∞ . An ideal I of the algebra B is a rotation invariant ideal if it is a proper (nonzero) closed ideal of B such that, whenever f is a function in I and λ is a point in ∂D , the function f_λ is also in I .

In this paper we discuss the existence of rotation invariant ideals of subalgebras of L^∞ containing H^∞ (the space of bounded analytic functions on D). Let C denote the space of continuous, complex valued functions on ∂D . It is known [7] that a closed subalgebra of L^∞ properly containing H^∞ must contain $H^\infty + C$. Using a theorem of Axler [1] we construct rotation invariant ideals of every closed subalgebra of L^∞ properly containing H^∞ .

The ideals $z^n H^\infty$ are the obvious rotation invariant ideals of H^∞ . In [5], S. Power asked whether there are any others. Using results about function algebras, we shall show that there exist rotation invariant ideals other than $z^n H^\infty$.

Let QC denote the largest C^* algebra contained in $H^\infty + C$ and $QA = QC \cap H^\infty$. In [9], many analogues of known theorems for the disc algebra A are proven for QA . We shall use these to describe rotation invariant ideals in QA and QC .

Throughout this paper many identifications are made. The set of all nonzero multiplicative linear functionals of a closed subalgebra B of L^∞ is called the maximal ideal space of B and is denoted $M(B)$. With the weak- $*$ topology, $M(B)$ is a compact Hausdorff space. There is one-to-one correspondence between maximal ideals of B and kernels of elements of $M(B)$. Each element in $M(H^\infty)$ has a unique norm preserving extension to a linear functional on L^∞ . Thus we identify $M(B)$ with a closed subset of $M(H^\infty)$. We think of D as a subset of $M(H^\infty)$. We shall identify a function in L^∞ with its harmonic extension to D . We also identify L^∞ with $C(M(L^\infty))$, the space of continuous functions on $M(L^\infty)$ and QC with $C(M(QC))$.

I thank S. Axler and T. Wolff for valuable discussions.

Received by the editors September 17, 1984.

1980 *Mathematics Subject Classification.* Primary 46H10, 46J15, 46J20.

¹Partially supported by a grant from the National Sciences Foundation.

I. Examples of rotation invariant ideals in subalgebras of L^∞ containing QC . Let E be a measurable subset of the unit circle. For a point of ∂D , let $E_\lambda = \{\lambda z : z \in E\}$. The set E is said to be permanently positive (PP) if the intersection of any finite number of translates of E has positive measure. W. Rudin [6] has shown the existence of PP sets and of a measurable set E such that E and $\partial D \sim E$ are permanently positive.

Suppose that E is a PP set. Let $F = \partial D \sim E$. For any finite set of points $\{\lambda_1, \dots, \lambda_n\}$ in ∂D , and L^∞ functions f_1, \dots, f_n , we have

$$\|1 - (f_1 X_{F_{\lambda_1}} + \dots + f_n X_{F_{\lambda_n}})\| \geq 1.$$

Therefore, the (closed) ideal I generated by $\{X_{F_\lambda} : \lambda \in \partial D\}$ is a rotation invariant ideal in L^∞ .

In [1] S. Axler has shown that if f is a function in L^∞ , then there exists a Blaschke product b such that bf is a function in $H^\infty + C$. In Lemma 1 below, we use this theorem to give examples of rotation invariant ideals in closed subalgebras of L^∞ containing $H^\infty + C$.

LEMMA 1. *Let B be a closed subalgebra of L^∞ containing $H^\infty + C$. Then there exists a rotation invariant ideal I of B .*

PROOF. Let E be a PP set and $F = \partial D \sim E$. By Axler's Theorem [1], there exists a Blaschke product b such that $bX_F \in H^\infty + C$. Therefore, $(bX_F)_\lambda = b_\lambda X_{F_{\lambda}}$ is a function in $H^\infty + C$ (hence B) for every λ in ∂D . Consider the closed ideal I of B generated by all rotates of bX_F . Since E is a PP set, the ideal generated by all rotates of X_F is a rotation invariant ideal in L^∞ . Therefore, there exists a maximal ideal in L^∞ containing it. Let x denote the element of $M(L^\infty)$ corresponding to the maximal ideal. Then

$$x((bX_F)_\lambda) = x(b_\lambda)x(X_{F_{\lambda}}) = 0$$

for all λ in ∂D . Hence I is a proper ideal of B . Since the ideal I is closed and $I_\lambda = I$ for all λ in ∂D , the proof is complete.

Let $x \in M(L^\infty)$ (or $M(QC)$) and $\lambda \in \partial D$. Then x_λ denotes the element of $M(L^\infty)$ (or $M(QC)$) defined by $x_\lambda(f) = x(f_\lambda)$ for all $f \in L^\infty$ (all $f \in QC$). The zero set of a function f is denoted $Z(f)$.

THEOREM 1. *An ideal I of L^∞ is rotation invariant if and only if there exists a closed, rotation invariant, nowhere dense set $\hat{F} \subset M(L^\infty)$ with $I = \{f \in L^\infty : f|_{\hat{F}} \equiv 0\}$. Furthermore, $\hat{F} = \bigcap \{Z(f) : f \in I\}$.*

PROOF. Let I be a rotation invariant ideal of L^∞ . Since I is a closed ideal and $M(L^\infty)$ is a compact Hausdorff space, we must have [8, p. 32] $I = \{f \in L^\infty : f|_{\hat{F}} = 0\}$ where $\hat{F} = \bigcap \{Z(f) : f \in I\}$. Let $x \in \hat{F}$ and $\lambda \in \partial D$. If f is a function in I , then f_λ is also in I . Therefore, $x_\lambda(f) = x(f_\lambda) = 0$. Thus if $x \in \hat{F}$, then $x_\lambda \in \hat{F}$ and \hat{F} is rotation invariant. It remains to show that \hat{F} is a nowhere dense subset of $M(L^\infty)$.

If \hat{F} were not nowhere dense, there would exist a closed and open set \hat{E} contained in \hat{F} . Let E be a measurable subset of ∂D of positive measure such that

$$\hat{E} = \{x \in M(L^\infty) : x(X_E) = 1\}.$$

Note that \hat{E}_λ is contained in \hat{F} for every λ in ∂D .

Let f be a (nonzero) nonnegative function in I . For any $x \in \hat{F}$,

$$x((f \cdot X_E)_\lambda) = x(f)x(X_{E_\lambda}) = 0.$$

If $x \in M(L^\infty) \sim \hat{F}$, then $x_\lambda \in M(L^\infty) \sim \hat{E}$. Thus $x(X_{E_\lambda}) = 0$. Therefore, for any λ in ∂D the function g defined by

$$g(\lambda, t) = (fX_{E_\lambda})(t)$$

is zero (a.e.). By the Fubini Theorem, we have

$$\begin{aligned} 0 &= \int \int g(\lambda, t) d(\lambda x t) = \int \int f(t)X_E(\lambda t) d\lambda dt \\ &= \int \int f(t)X_{E_\lambda}(t) d\lambda dt = \int m(E_\lambda)f(t) dt = m(E) \int f(t) dt > 0, \end{aligned}$$

a contradiction. Thus \hat{F} is nowhere dense.

Now suppose there exists a closed, rotation invariant set $\hat{F} \subset M(L^\infty)$ such that $I = \{f \in L^\infty: f|_{\hat{F}} = 0\}$. Let f be a function in I and λ be a point in ∂D . For any $x \in \hat{F}$ we have $x(f_\lambda) = x_\lambda(f) = 0$. Thus, $f_\lambda|_{\hat{F}} = 0$ and I is a rotation invariant ideal.

COROLLARY 1. *An ideal I is a closed, rotation invariant ideal of QC if and only if $I = \{f \in QC: f|_{\hat{F}} = 0\}$ for some closed, rotation invariant, nowhere dense set $\hat{F} \subset M(QC)$. Furthermore, $\hat{F} = \bigcap \{Z(f): f \in I\}$.*

We shall only show that such a set \hat{F} must be nowhere dense. The rest of the proof is the same as the preceding proof and is omitted.

In what follows $\Gamma: M(L^\infty) \rightarrow M(QC)$ denotes the restriction map of $M(L^\infty)$ onto $M(QC)$. The map Γ is continuous.

PROOF OF COROLLARY 1. Suppose \mathcal{O} is an open subset of $M(QC)$ and $\mathcal{O} \subseteq \hat{F}$. Then $\Gamma^{-1}(\mathcal{O})$ is an open set contained in the closed set $\Gamma^{-1}(\hat{F})$. Furthermore, if q is a function in I , then $q|_{\Gamma^{-1}(\hat{F})} = 0$. Therefore, the ideal J defined by $J = \overline{L}^\infty$ is a rotation invariant ideal in L^∞ . Then we have

$$\Gamma^{-1}(\mathcal{O}) \subseteq \Gamma^{-1}(\hat{F}) \subset \bigcap \{Z(q): q \in I\} = \bigcap \{Z(f): f \in J\}.$$

Let $\hat{G} = \bigcap \{Z(f): f \in J\}$. Then \hat{G} is somewhere dense, which contradicts Theorem 1. Therefore, \hat{F} is nowhere dense.

If I is a rotation invariant ideal L^∞ or QC , the set \hat{F} above is rotation invariant and therefore cannot consist of a single point. Thus, there are no prime rotation invariant ideals.

II. Rotation invariant ideals in H^∞ and QA . The algebras H^∞ and QA have the obvious rotation invariant ideals $z^n H^\infty$ and $z^n QA$, respectively, for some positive integer n . Theorem 2 shows that these are the only finitely generated rotation invariant ideals.

In the next lemma, which will be used to prove Theorem 2, we use the fact that $M(H^\infty + C) = M(H^\infty) \sim D$.

LEMMA 2. *Let I be a finitely generated rotation invariant ideal in H^∞ . Then $I = z^n H^\infty$ for some positive integer n if and only if there exists no $x \in M(H^\infty) \sim D$ containing I in its kernel.*

PROOF. Suppose $I = z^n H^\infty$ for some positive integer n . Since \bar{z} is an element of $H^\infty + C$, one sees easily that $|x(z)| = 1$ for any $x \in M(H^\infty) \sim D$. Therefore, I is not contained in the kernel of any element of $M(H^\infty) \sim D$.

Let $\{f_n : n = 1, 2, 3, \dots, k\}$ be a set of generators for I . There exists a maximal ideal containing I . Our assumptions imply that I is contained in the single maximal ideal $\{f : f(0) = 0\}$. Let

$$n = \max\{j : z^{-j} f_m \in H^\infty \text{ for all } m\}.$$

If $\bar{z}^n I$ were a proper ideal of H^∞ , then it would obviously be a rotation invariant ideal. Let x be an element of $M(H^\infty) \sim D$ containing $\bar{z}^n I$ in its kernel. Then x also contains I in its kernel. As before, the only maximal ideal containing $\bar{z}^n I$ must be $\{f : f(0) = 0\}$. This implies that, for any positive integer m , $\bar{z}(\bar{z}^n f_m) = \bar{z}^{n+1} f_m$ is an H^∞ function, contradicting the maximality of n . Therefore, $\bar{z}^n I = H^\infty$, as desired.

We note that the preceding proof together with the fact that $M(QC) = M(QA) \sim D$ can be used to obtain the following lemma.

LEMMA 3. *Let I be a finitely generated rotation invariant ideal in QA . Then $I = z^n QA$ for some positive integer n if and only if there exists no $x \in M(QC)$ containing I in its kernel.*

In what follows, $C(\lambda : f)$ denotes the cluster set of f at λ . If $R(\lambda)$ is a radius terminating at λ , then $C(R(\lambda) : f)$ denotes the cluster set of f along $R(\lambda)$.

THEOREM 2. *If I is a finitely generated rotation invariant ideal in H^∞ , then $I = z^n H^\infty$ for some positive integer n .*

PROOF. Let I be generated by f_1, \dots, f_m and suppose that $I \neq z^n H^\infty$ for any positive integer n . By Lemma 2, there exists a multiplicative linear functional x in $M(H^\infty) \sim D$ such that $I \subseteq \ker x$. Let g be the functional defined by

$$g(z) = |f_1(z)| + \dots + |f_m(z)|.$$

Then g is a continuous function on D . Let $f_{m+1}(z) = z$. By assumption, I is a rotation invariant ideal. Therefore, for each λ in ∂D we have $I \subseteq \ker x_\lambda$. By the Corona Theorem, for each positive integer k and each λ in ∂D , there is a point $z_{k,\lambda}$ of D such that

$$z_{k,\lambda} \in \bigcap_{j=1}^{m+1} \{y \in M(H^\infty) : |x_\lambda(f_j) - y(f_j)| < 1/k\}.$$

Therefore, $0 \in C(\lambda : g)$ for all $\lambda \in \partial D$. By Collingwood's maximality theorem [2], there exists λ_0 such that $0 \in C(R(\lambda_0) : g)$. Let x_0 be a point in the closure of a radial sequence such that $x_0(f_j) = 0$ for $j = 1, 2, \dots, m$. Let \tilde{g} be a nonzero function in the ideal I . Then for any $\lambda \in \partial D$, the function \tilde{g}_λ is also in the ideal I . Since $x_0(f) = 0$ for any f in the ideal I we have that $x_0(\tilde{g}_\lambda) = 0$ for all $\lambda \in \partial D$. By Fatou's Theorem, the (nonzero) function \tilde{g} must have radial limit zero almost everywhere, a contradiction. Thus $I = z^n H^\infty$ for some positive integer n .

Replacing H^∞ by QA in the proof above and using Lemma 3 in place of Lemma 2 establishes the following corollary.

COROLLARY 2. *If I is a finitely generated rotation invariant ideal in QA , then $I = z^n QA$ for some positive integer n .*

A closed set $F \subseteq M(QC)$ is called a peak interpolation set in the weak sense for QA if whenever g is a continuous function on F , then there is a function f in QA such that $f(t) = g(t)$ when $t \in F$ and $\|f\|_\infty = \max_{s \in F} |g(s)|$.

THEOREM 3. *There exist rotation invariant ideals in QA other than $z^n QA$.*

PROOF. Let I be a rotation invariant ideal in QC . By Corollary 1, there exists a closed, nowhere dense, rotation invariant set $\hat{F} \subseteq M(QC)$ such that $I = \{f \in QC: f|_{\hat{F}} = 0\}$. Let $f \in I$ be a nonzero function. Let $t \in M(QC)$ be such that $f(t) \neq 0$. Then $\hat{F} \cup \{t\}$ is a closed, nowhere dense set in $M(QC)$. T. Wolff [10] has shown that a closed nowhere dense subset of $M(QC)$ is a peak interpolation set in the weak sense for QA . Thus, there exists a QA function q such that $q|_{\hat{F} \cup \{t\}} = f|_{\hat{F} \cup \{t\}}$. Consider the (closed) ideal generated by $\{q_\lambda: \lambda \in \partial D\}$. If $x \in \hat{F}$, then $x_\lambda \in \hat{F}$ for all $\lambda \in \partial D$. Thus $x(q_\lambda) = x_\lambda(q) = 0$ for all $\lambda \in \partial D$. Therefore, for any finite set of functions $\{f_1, f_2, \dots, f_n\}$ in QA , we have

$$\|1 - (f_1 q_{\lambda_1} + \dots + f_n q_{\lambda_n})\| \geq 1.$$

Lemma 3 implies that I is a rotation invariant ideal in QA and $I \neq z^n H$ for any positive integer n .

By replacing x by some element of $\Gamma^{-1}(x)$ and choosing the subset above from H^∞ rather than from QA we obtain

COROLLARY 3. *There exists a rotation invariant ideal I in H^∞ such that $I \neq z^n H^\infty$ for any positive integer n .*

REFERENCES

1. S. Axler, *Factorization of L^∞ functions*, Ann. of Math. (2) **106** (1977), 567–572.
2. E. F. Collingwood, *On sets of maximum indetermination of analytic functions*, Math. Z. **67** (1957), 377–396.
3. T. W. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N.J., 1969.
4. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
5. S. C. Power, *On ideals of nest subalgebras of C^* -algebras*, Proc. London Math. Soc. (3) **50** (1985), 314–332.
6. W. Rudin, *Invariant means on L^∞* , Studia Math. **44** (1972), 219–227.
7. D. E. Sarason, *Algebras of functions on the unit circle*, Bull. Amer. Math. Soc. **79** (1973), 286–300.
8. E. L. Stout, *The theory of uniform algebras*, Bogden & Quigley, New York, 1971.
9. T. Wolff, *Two algebras of bounded functions*, Duke Math. J. **49** (1982), 321–328.
10. —, *Some theorems on vanishing mean oscillation*, Thesis, Univ. of California at Berkeley, 1979.

DEPARTMENT OF MATHEMATICS, BUCKNELL UNIVERSITY, LEWISBURG, PENNSYLVANIA 17837