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ROTATION INVARIANT IDEALS IN SUBALGEBRAS OF L^∞

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ABSTRACT. In this paper, it is shown that the only (nontrivial) finitely generated rotation invariant ideals in H^∞ are $z^n H^\infty$ for some positive integer n. Using results about function algebras, it is shown that other rotation invariant ideals exist. Rotation invariant ideals of other subalgebras of L^∞ are also studied.

Let D and ∂D be the open unit disc and its boundary, respectively, in the complex plane C. Let L^∞ be the Banach algebra of all essentially bounded Lebesgue measurable functions on ∂D with the essential supremum norm. If f is a function in L^∞ and λ is a point in ∂D, let f_λ denote the L^∞ function defined by $f_\lambda(z) = f(\lambda z)$. Let B denote a closed subalgebra of L^∞. An ideal I of the algebra B is a rotation invariant ideal if it is a proper (nonzero) closed ideal of B such that, whenever f is a function in I and λ is a point in ∂D, the function f_λ is also in I.

In this paper we discuss the existence of rotation invariant ideals of subalgebras of L^∞ containing H^∞ (the space of bounded analytic functions on D). Let C denote the space of continuous, complex valued functions on ∂D. It is known [**7**] that a closed subalgebra of L^∞ properly containing H^∞ must contain $H^\infty + C$. Using a theorem of Axler [**1**] we construct rotation invariant ideals of every closed subalgebra of L^∞ properly containing H^∞.

The ideals $z^n H^\infty$ are the obvious rotation invariant ideals of H^∞. In [**5**], S. Power asked whether there are any others. Using results about function algebras, we shall show that there exist rotation invariant ideals other than $z^n H^\infty$.

Let QC denote the largest C-* algebra contained in $H^\infty + C$ and $QA = QC \cap H^\infty$. In [**9**], many analogues of known theorems for the disc algebra A are proven for QA. We shall use these to describe rotation invariant ideals in QA and QC.

Throughout this paper many identifications are made. The set of all nonzero multiplicative linear functionals of a closed subalgebra B of L^∞ is called the maximal ideal space of B and is denoted $M(B)$. With the weak-* topology, $M(B)$ is a compact Hausdorff space. There is one-to-one correspondence between maximal ideals of B and kernels of elements of $M(B)$. Each element in $M(H^\infty)$ has a unique norm preserving extension to a linear functional on L^∞. Thus we identify $M(B)$ with a closed subset of $M(H^\infty)$. We think of D as a subset of $M(H^\infty)$. We shall identify a function in L^∞ with its harmonic extension to D. We also identify L^∞ with $C(M(L^\infty))$, the space of continuous functions on $M(L^\infty)$ and QC with $C(M(QC))$.

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I. Examples of rotation invariant ideals in subalgebras of L^∞ containing QC. Let E be a measurable subset of the unit circle. For a point of ∂D, let $E_\lambda = \{\lambda z: z \in E\}$. The set E is said to be permanently positive (PP) if the intersection of any finite number of translates of E has positive measure. W. Rudin [**6**] has shown the existence of PP sets and of a measurable set E such that E and $\partial D \sim E$ are permanently positive.

Suppose that E is a PP set. Let $F = \partial D \sim E$. For any finite set of points $\{\lambda_1, \ldots, \lambda_n\}$ in ∂D, and L^∞ functions $f_1, \ldots, f_n$, we have

$$\|1 - (f_1 X_{F_{\lambda_1}} + \cdots + f_n X_{F_{\lambda_n}})\| \geq 1.$$

Therefore, the (closed) ideal I generated by $\{X_{F_\lambda}: \lambda \in \partial D\}$ is a rotation invariant ideal in L^∞.

In [**1**] S. Axler has shown that if f is a function in L^∞, then there exists a Blaschke product b such that bf is a function in $H^\infty + C$. In Lemma 1 below, we use this theorem to give examples of rotation invariant ideals in closed subalgebras of L^∞ containing $H^\infty + C$.

LEMMA 1. *Let B be a closed subalgebra of L^∞ containing $H^\infty + C$. Then there exists a rotation invariant ideal I of B.*

PROOF. Let E be a PP set and $F = \partial D \sim E$. By Axler's Theorem [**1**], there exists a Blaschke product b such that $bX_F \in H^\infty + C$. Therefore, $(bX_F)_\lambda = b_\lambda X_{F_{\bar\lambda}}$ is a function in $H^\infty + C$ (hence B) for every λ in ∂D. Consider the closed ideal I of B generated by all rotates of bX_F. Since E is a PP set, the ideal generated by all rotates of X_F is a rotation invariant ideal in L^∞. Therefore, there exists a maximal ideal in L^∞ containing it. Let x denote the element of $M(L^\infty)$ corresponding to the maximal ideal. Then

$$x((bX_F)_\lambda) = x(b_\lambda)x(X_{F_{\bar\lambda}}) = 0$$

for all λ in ∂D. Hence I is a proper ideal of B. Since the ideal I is closed and $I_\lambda = I$ for all λ in ∂D, the proof is complete.

Let $x \in M(L^\infty)$ (or $M(QC)$) and $\lambda \in \partial D$. Then x_λ denotes the element of $M(L^\infty)$ (or $M(QC)$) defined by $x_\lambda(f) = x(f_\lambda)$ for all $f \in L^\infty$ (all $f \in QC$). The zero set of a function f is denoted $Z(f)$.

THEOREM 1. *An ideal I of L^∞ is rotation invariant if and only if there exists a closed, rotation invariant, nowhere dense set $\hat F \subset M(L^\infty)$ with $I = \{f \in L^\infty: f|\hat F \equiv 0\}$. Furthermore, $\hat F = \bigcap\{Z(f): f \in I\}$.*

PROOF. Let I be a rotation invariant ideal of L^∞. Since I is a closed ideal and $M(L^\infty)$ is a compact Hausdorff space, we must have [**8**, p. 32] $I = \{f \in L^\infty: f|\hat F = 0\}$ where $\hat F = \bigcap\{Z(f): f \in I\}$. Let $x \in \hat F$ and $\lambda \in \partial D$. If f is a function in I, then f_λ is also in I. Therefore, $x_\lambda(f) = x(f_\lambda) = 0$. Thus if $x \in \hat F$, then $x_\lambda \in \hat F$ and $\hat F$ is rotation invariant. It remains to show that $\hat F$ is a nowhere dense subset of $M(L^\infty)$.

If $\hat F$ were not nowhere dense, there would exist a closed and open set $\hat E$ contained in $\hat F$. Let E be a measurable subset of ∂D of positive measure such that

$$\hat E = \{x \in M(L^\infty): x(X_E) = 1\}.$$

Note that $\hat E_\lambda$ is contained in $\hat F$ for every λ in ∂D.

Let f be a (nonzero) nonnegative function in I. For any $x \in \hat{F}$,

$$x((f \cdot X_E)_\lambda) = x(f)x(X_{E_{\bar{\lambda}}}) = 0.$$

If $x \in M(L^\infty) \sim \hat{F}$, then $x_\lambda \in M(L^\infty) \sim \hat{E}$. Thus $x(X_{E_{\bar{\lambda}}}) = 0$. Therefore, for any λ in ∂D the function g defined by

$$g(\lambda, t) = (fX_{E_{\bar{\lambda}}})(t)$$

is zero (a.e.). By the Fubini Theorem, we have

$$\begin{aligned} O &= \int g(\lambda, t)\, d(\lambda x t) = \int\int f(t) X_E(\lambda t)\, d\lambda\, dt \\ &= \int\int f(t) X_{E_{\bar{t}}}(\lambda)\, d\lambda\, dt = \int m(E_{\bar{t}}) f(t)\, dt = m(E) \int f(t)\, dt > 0, \end{aligned}$$

a contradiction. Thus $\hat{F}$ is nowhere dense.

Now suppose there exists a closed, rotation invariant set $\hat{F} \subset M(L^\infty)$ such that $I = \{f \in L^\infty : f|\hat{F} = 0\}$. Let f be a function in I and λ be a point in ∂D. For any $x \in \hat{F}$ we have $x(f_\lambda) = x_\lambda(f) = 0$. Thus, $f_\lambda|\hat{F} = 0$ and I is a rotation invariant ideal.

COROLLARY 1. *An ideal I is a closed, rotation invariant ideal of QC if and only if $I = \{f \in QC : f|\hat{F} = 0\}$ for some closed, rotation invariant, nowhere dense set $\hat{F} \subset M(QC)$. Furthermore, $\hat{F} = \bigcap\{Z(f) : f \in I\}$.*

We shall only show that such a set $\hat{F}$ must be nowhere dense. The rest of the proof is the same as the preceding proof and is omitted.

In what follows $\Gamma : M(L^\infty) \to M(QC)$ denotes the restriction map of $M(L^\infty)$ onto $M(QC)$. The map Γ is continuous.

PROOF OF COROLLARY 1. Suppose $\mathcal{O}$ is an open subset of $M(QC)$ and $\mathcal{O} \subseteq \hat{F}$. Then $\Gamma^{-1}(\mathcal{O})$ is an open set contained in the closed set $\Gamma^{-1}(\hat{F})$. Furthermore, if q is a function in I, then $q|\Gamma^{-1}(\hat{F}) = 0$. Therefore, the ideal J defined by $J = \overline{IL^\infty}$ is a rotation invariant ideal in L^∞. Then we have

$$\Gamma^{-1}(\mathcal{O}) \subseteq \Gamma^{-1}(\hat{F}) \subset \bigcap\{Z(q) : q \in I\} = \bigcap\{Z(f) : f \in J\}.$$

Let $\hat{G} = \bigcap\{Z(f) : f \in J\}$. Then $\hat{G}$ is somewhere dense, which contradicts Theorem 1. Therefore, $\hat{F}$ is nowhere dense.

If I is a rotation invariant ideal L^∞ or QC, the set $\hat{F}$ above is rotation invariant and therefore cannot consist of a single point. Thus, there are no prime rotation invariant ideals.

II. Rotation invariant ideals in H^∞ and QA. The algebras H^∞ and QA have the obvious rotation invariant ideals $z^n H^\infty$ and $z^n QA$, respectively, for some positive integer n. Theorem 2 shows that these are the only finitely generated rotation invariant ideals.

In the next lemma, which will be used to prove Theorem 2, we use the fact that $M(H^\infty + C) = M(H^\infty) \sim D$.

LEMMA 2. *Let I be a finitely generated rotation invariant ideal in H^{∞}. Then $I = z^n H^{\infty}$ for some positive integer n if and only if there exists no $x \in M(H^{\infty}) \sim D$ containing I in its kernel.*

PROOF. Suppose $I = z^n H^{\infty}$ for some positive integer n. Since $\overline{z}$ is an element of $H^{\infty} + C$, one sees easily that $|x(z)| = 1$ for any $x \in M(H^{\infty}) \sim D$. Therefore, I is not contained in the kernel of any element of $M(H^{\infty}) \sim D$.

Let $\{f_n: n = 1, 2, 3, \ldots, k\}$ be a set of generators for I. There exists a maximal ideal containing I. Our assumptions imply that I is contained in the single maximal ideal $\{f: f(0) = 0\}$. Let

$$n = \max\{j: z^{-j} f_m \in H^{\infty} \text{ for all } m\}.$$

If $\overline{z}^{\,n} I$ were a proper ideal of H^{∞}, then it would obviously be a rotation invariant ideal. Let x be an element of $M(H^{\infty}) \sim D$ containing $\overline{z}^{\,n} I$ in its kernel. Then x also contains I in its kernel. As before, the only maximal ideal containing $\overline{z}^{\,n} I$ must be $\{f: f(0) = 0\}$. This implies that, for any positive integer m, $\overline{z}(\overline{z}^{\,n} f_m) = \overline{z}^{\,n+1} f_m$ is an H^{∞} function, contradicting the maximality of n. Therefore, $\overline{z}^{\,n} I = H^{\infty}$, as desired.

We note that the preceding proof together with the fact that $M(QC) = M(QA) \sim D$ can be used to obtain the following lemma.

LEMMA 3. *Let I be a finitely generated rotation invariant ideal in QA. Then $I = z^n QA$ for some positive integer n if and only if there exists no $x \in M(QC)$ containing I in its kernel.*

In what follows, $C(\lambda : f)$ denotes the cluster set of f at λ. If $R(\lambda)$ is a radius terminating at λ, then $C(R(\lambda) : f)$ denotes the cluster set of f along $R(\lambda)$.

THEOREM 2. *If I is a finitely generated rotation invariant ideal in H^{∞}, then $I = z^n H^{\infty}$ for some positive integer n.*

PROOF. Let I be generated by $f_1, \ldots, f_m$ and suppose that $I \neq z^n H^{\infty}$ for any positive integer n. By Lemma 2, there exists a multiplicative linear functional x in $M(H^{\infty}) \sim D$ such that $I \subseteq \ker x$. Let g be the functional defined by

$$g(z) = |f_1(z)| + \cdots + |f_m(z)|.$$

Then g is a continuous function on D. Let $f_{m+1}(z) = z$. By assumption, I is a rotation invariant ideal. Therefore, for each λ in ∂D we have $I \subseteq \ker x_{\lambda}$. By the Corona Theorem, for each positive integer k and each λ in ∂D, there is a point $z_{k,\lambda}$ of D such that

$$z_{k,\lambda} \in \bigcap_{j=1}^{m+1} \{y \in M(H^{\infty}): |x_{\lambda}(f_j) - y(f_j)| < 1/k\}.$$

Therefore, $0 \in C(\lambda : g)$ for all $\lambda \in \partial D$. By Collingwood's maximality theorem [**2**], there exists λ_0 such that $0 \in C(R(\lambda_0) : g)$. Let x_0 be a point in the closure of a radial sequence such that $x_0(f_j) = 0$ for $j = 1, 2, \ldots, m$. Let $\tilde{g}$ be a nonzero function in the ideal I. Then for any $\lambda \in \partial D$, the function $\tilde{g}_{\lambda}$ is also in the ideal I. Since $x_0(f) = 0$ for any f in the ideal I we have that $x_0(\tilde{g}_{\lambda}) = 0$ for all $\lambda \in \partial D$. By Fatou's Theorem, the (nonzero) function $\tilde{g}$ must have radial limit zero almost everywhere, a contradiction. Thus $I = z^n H^{\infty}$ for some positive integer n.

Replacing H^∞ by QA in the proof above and using Lemma 3 in place of Lemma 2 establishes the following corollary.

COROLLARY 2. *If I is a finitely generated rotation invariant ideal in QA, then $I = z^n QA$ for some positive integer n.*

A closed set $F \subseteq M(QC)$ is called a peak interpolation set in the weak sense for QA if whenever g is a continuous function on F, then there is a function f in QA such that $f(t) = g(t)$ when $t \in F$ and $\|f\|_\infty = \max_{s\in F} |g(s)|$.

THEOREM 3. *There exist rotation invariant ideals in QA other than $z^n QA$.*

PROOF. Let I be a rotation invariant ideal in QC. By Corollary 1, there exists a closed, nowhere dense, rotation invariant set $\hat{F} \subseteq M(QC)$ such that $I = \{f \in QC: f|\hat{F} = 0\}$. Let $f \in I$ be a nonzero function. Let $t \in M(QC)$ be such that $f(t) \neq 0$. Then $\hat{F} \cup \{t\}$ is a closed, nowhere dense set in $M(QC)$. T. Wolff [**10**] has shown that a closed nowhere dense subset of $M(QC)$ is a peak interpolation set in the weak sense for QA. Thus, there exists a QA function q such that $q|\hat{F} \cup \{t\} = f|\hat{F} \cup \{t\}$. Consider the (closed) ideal generated by $\{q_\lambda : \lambda \in \partial D\}$. If $x \in \hat{F}$, then $x_\lambda \in \hat{F}$ for all $\lambda \in \partial D$. Thus $x(q_\lambda) = x_\lambda(q) = 0$ for all $\lambda \in \partial D$. Therefore, for any finite set of functions $\{f_1, f_2, \ldots, f_n\}$ in QA, we have

$$\|1 - (f_1 q_{\lambda_1} + \cdots + f_n q_{\lambda_n})\| \geq 1.$$

Lemma 3 implies that I is a rotation invariant ideal in QA and $I \neq z^n H$ for any positive integer n.

By replacing x by some element of $\Gamma^{-1}(x)$ and choosing the subset above from H^∞ rather than from QA we obtain

COROLLARY 3. *There exists a rotation invariant ideal I in H^∞ such that $I \neq z^n H^\infty$ for any positive integer n.*

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