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ON SUCCESSIVE COEFFICIENTS OF UNIVALENT FUNCTIONS

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ABSTRACT. Let $f(z) \in S$, that is, $f(z)$ is analytic and univalent in the unit disk $|z| < 1$, normalized by $f(0) = f'(0) - 1 = 0$. Let p be real and

$$\{f(z)/z\}^p = 1 + \sum_{n=1}^{\infty} D_n(p) z^n.$$

Lucas proved that

$$\big| |D_n(p)| - |D_{n+1}(p)| \big| \le A n^{(t(p)-1)/2} \log^{3/2} n, \qquad n = 2, 3, \ldots,$$

for some absolute constant A and $t(p) = (2\sqrt{p} - 1)^2$. In this paper we improve $t(p)$ as follows:

$$T(p) = \frac{4p-1}{2p + t(p)} t(p).$$

Let the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$, and

$$F_p(z) = \left\{ \frac{f(z)}{z} \right\}^p = 1 + \sum_{n=1}^{\infty} D_n(p) z^n.$$

It is a very interesting problem to find a best possible number $t(p)$ for which the inequality

$$\big| |D_n(p)| - |D_{n-1}(p)| \big| \le A n^{(t(p)-1)/2}$$

holds, where A is an absolute constant.

This problem was first studied by Goluzin. In 1963 Hayman obtained a precise result $t(1) = 1$. In 1956 the author [**2**] proved that $t(p) = 2p - 1$ $(0 < p < 1)$ for $f \in S^*$. In the general case, the better result $t(p) = (2\sqrt{p} - 1)^2$ $(\frac{1}{4} < p < 1)$ is due to Lucas [**1**].

THEOREM. *Let $f(z) \in S$, and $F_p(z) = \{f(z)/z\}^p = 1 + \sum_{n=1}^{\infty} D_n(p) z^n$. Then for every $p \in (\frac{1}{4}, 1]$, we have*

$$\big| |D_n(p)| - |D_{n-1}(p)| \big| \le A n^{(T(p)-1)/2} \log^{3/2} n, \qquad n = 2, 3, \ldots,$$

where A is an absolute constant, and

$$T(p) = \frac{4p-1}{2p + t(p)} t(p), \qquad t(p) = (2\sqrt{p} - 1)^2.$$

This estimate is obviously better than Lucas's because $(4p-1)/(2p + t(p)) < 1$ for $\frac{1}{4} < p < 1$. Note that if $p = \frac{1}{2}$ then $zF_p(z^2)$ is an odd univalent function, and so on.

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1. Lemmas. We require some lemmas. In the following, $A, A_1, A_2, \dots$ will denote some absolute constants.

LEMMA 1 [**4**]. *Let $f(z) \in S$. Then*

$$\int_0^{2\pi} \left| r\frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|^2 d\theta \le \frac{A}{1-r} \log\frac{1}{1-r}, \qquad \frac{1}{2} \le r < 1. \tag{1.1}$$

LEMMA 2. *Let $f(z) \in S$ and $F_p(z) = \{f(z)/z\}^p = 1 + \sum_{n=1}^{\infty} D_n(p) z^n$. Let ρ be given such that $\max_{|z|=\rho} |f(z)| = |f(\rho)|$. Then for every $p \in (\frac{1}{4}, 1]$ we have*

$$|re^{i\theta} - \rho|^2 |F_p(re^{i\theta})|^2 \le \begin{cases} 4r(1-\rho)^{2p}/(1-r)^{4p}, & 0 < \rho < r < 1, \\ 2^{3-2\sqrt{p}} \rho^{2-2p} r^{-t(p)/2}/(1-r)(1-\rho)^{t(p)}, & 0 < r \le \rho < 1, \end{cases} \tag{1.2}$$

where $t(p) = (2\sqrt{p} - 1)^2$.

PROOF. By Goluzin's inequality [**5**]

$$\begin{aligned} &\left(|1 - z\rho| \left|\frac{z-\rho}{z\rho}\right|\right)^{2x_1x_2} (1-\rho^2)^{x_1^2} (1-|z|^2)^{x_2^2} \\ &\qquad \le \left|\frac{1}{f(z)} - \frac{1}{f(\rho)}\right|^{2x_1x_2} \left|\frac{z^2 f'(z)}{f^2(z)}\right|^{x_2^2} \left|\frac{\rho^2 f'(\rho)}{f^2(\rho)}\right|^{x_1^2}, \end{aligned} \tag{1.3}$$

where x_1, x_2 are real numbers.

The following inequalities are known:

$$|\varsigma f'(\varsigma)/f(\varsigma)| \le (1+|\varsigma|)/(1-|\varsigma|), \qquad |\varsigma| < 1, \tag{1.4}$$

$$t^{-1}(1-t)^2 |f(te^{i\theta})| \le s^{-1}(1-s)^2 |f(se^{i\theta})|, \qquad 0 < s \le t < 1. \tag{1.5}$$

From inequality (1.5) and the hypothesis that $0 < \rho \le r < 1$ and $\max_{|z|=\rho}|f(z)| = |f(\rho)|$, it is easy to show

$$\begin{aligned} \frac{1}{|f(re^{i\theta})|} + \frac{1}{|f(p)|} &\le \frac{1}{|f(re^{i\theta})|} + \frac{1}{|f(\rho e^{i\theta})|} \\ &\le \frac{1}{|f(re^{i\theta})|} + \frac{r(1-\rho)^2}{\rho(1-r)^2|f(re^{i\theta})|} \le \frac{2r(1-p)^2}{\rho(1-r)^2|f(re^{i\theta})|}. \end{aligned} \tag{1.6}$$

We choose $x_1 = x_2 = \sqrt{p}$ in (1.3) and notice that $|z - \rho| \le |1 - \rho z|$ and $|z - \rho| \le 2r$. Then from (1.4), (1.5), and (1.6) it is not difficult to deduce the first inequality in (1.2).

The second inequality in (1.2) [**1**] can also be obtained by putting $x_1 = 2\sqrt{p} - 1$, $x_2 = 1$ in (1.3). Thus the proof is complete.

LEMMA 3. *With the above assumption, we have*

$$\begin{aligned} J(t) &= \int_0^{2\pi} |te^{i\theta} - \rho|^2 \left| t\frac{f'(te^{i\theta})}{f(te^{i\theta})} \right|^2 |f(te^{i\theta})|^{2p}\, d\theta \\ &\le \begin{cases} \dfrac{A}{(1-t)^2(1-\rho)^{t(p)}} \log\dfrac{1}{1-\rho}, & \dfrac{1}{2} < t \le \rho < 1, \\ \dfrac{A(1-\rho)^{2p}}{(1-t)^{4p+1}} \log\dfrac{1}{1-t}, & \dfrac{1}{2} < \rho \le t < 1. \end{cases} \end{aligned} \tag{1.7}$$

PROOF. The proof follows from Lemmas 1 and 2.

LEMMA 4. *With the above assumption, write*

$$\varphi(z) = (\rho - z)\left\{\frac{f(z)}{z}\right\}^p = \rho + \sum_{n=1}^{\infty}(\rho D_n(p) - D_{n-1}(p))z^n.$$

Then

$$I(r) = \int_0^{2\pi} |\varphi(re^{i\theta})|^2\, d\theta$$

$$\leq \begin{cases} \dfrac{A_1}{(1-\rho)^{t(p)}} \log^2 \dfrac{1}{1-\rho}, & \dfrac{1}{2} \leq r \leq \rho < 1, \\ A_2 \left\{ \dfrac{1}{(1-\rho)^{t(p)}} + \dfrac{(1-\rho)^{2p}}{(1-r)^{4p-1}} \right\} \log^2 \dfrac{1}{1-r}, & \dfrac{1}{2} \leq \rho \leq r \leq 1. \end{cases} \tag{1.8}$$

PROOF. Since

$$|z\varphi'(z)|^2 = p^2 \left| (\rho - z)\frac{zf'(z)}{f(z)}\left\{\frac{f(z)}{z}\right\}^p - (\rho - z)\left\{\frac{f(z)}{z}\right\}^p - \frac{z}{p}\left\{\frac{f(z)}{z}\right\}^p \right|^2$$

$$\leq A_3 \left(|\rho - z|^2 \left|\frac{zf'(z)}{f(z)}\right|^2 \cdot \left|\frac{f(z)}{z}\right|^{2p} + \left|\frac{f(z)}{z}\right|^{2p} \right)$$

and it is known that

$$\int_0^{2\pi} \left|\frac{f(te^{i\theta})}{te^{i\theta}}\right|^{4p} d\theta \leq \frac{A_4}{(1-t)^{4p-1}} \qquad (0 < t < 1),$$

for $p > \frac{1}{4}$, hence

$$\begin{aligned} I'(r) &= \frac{d}{dr}\int_0^{2\pi} |\varphi(re^{i\theta})|^2 d\theta = \frac{4}{r}\int_0^r \int_0^{2\pi} |\varphi'(te^{i\theta})|^2 t\, dt\, d\theta \\ &= \frac{4}{r}\left(\int_0^{1/2} + \int_{1/2}^r\right)\int_0^{2\pi} |\varphi'(te^{i\theta})|^2\, d\theta\, t\, dt \\ &\leq A_5 + A_6 \int_{1/2}^r t\, dt \left(\int_0^{2\pi} |te^{i\theta} - \rho|^2 \left| t\frac{f'(te^{i\theta})}{f(te^{i\theta})}\right|^2 |f_p(te^{i\theta})|^2\, d\theta \right) \\ &\quad + \int_{1/2}^r \frac{A_7 dt}{(1-t)^{4p-1}}. \end{aligned} \tag{1.9}$$

If $\frac{1}{2} < r \leq \rho < 1$, (1.9), together with the first inequality in (1.7), yields

$$I'(r) \leq \frac{A_8}{(1-r)(1-\rho)^{t(p)}} \log \frac{1}{1-r}, \quad \frac{1}{2} \leq r < \rho < 1.$$

Integrating both sides of the above inequality with respect to r from $\frac{1}{2}$ to ρ we obtain the first inequality in (1.8).

If $1 > r \geq \rho$ we write

$$\int_0^r dt \int_0^{2\pi} |te^{i\theta} - \rho|^2 \left| t\frac{f'}{f}\right|^2 |F_p|^2\, d\theta$$

$$= \left(\int_0^{1/2} + \int_{1/2}^{\rho} + \int_{\rho}^r\right)\int_0^{2\pi} |te^{i\theta} - \rho|^2 \left|t\frac{f'}{f}\right|^2 |F_p|^2\, d\theta\, dt.$$

By Lemma 3, we have

$$I'(r) \le A_9 \left(\frac{1}{(1-\rho)^{t(p)+1}} + \frac{(1-\rho)^{2p}}{(1-r)^{4p}} \right) \log \frac{1}{1-r}, \qquad \frac{1}{2} \le \rho < r < 1.$$

Again integrating both sides of the above inequality from ρ to r yields

$$\begin{aligned} I(r) &\le I(\rho) + A_{10} \left(\frac{r-\rho}{(1-\rho)^{t(p)+1}} + \frac{(1-\rho)^{2p}}{(1-r)^{4p-1}} \right) \log \frac{1}{1-r} \\ &\le I(\rho) + A_{10} \left(\frac{1}{(1-\rho)^{t(p)}} + \frac{(1-\rho)^{2p}}{(1-r)^{4p-1}} \right) \log \frac{1}{1-r}, \qquad \frac{1}{2} \le \rho \le r < 1. \end{aligned}$$

By the first inequality of (1.8), we get the second inequality in (1.8). Thus the lemma follows.

2. Proof of the Theorem. If $f \in S$, then the rotation $e^{-i\varphi} f(e^{i\varphi} z)$ also belongs to S. This ration does not change the magnitudes of the coefficients of the corresponding function $F_p(z)$. Thus for a fixed ρ with $0 < \rho < 1$, there is no loss of generality in supposing that $\max_{|z|=\rho} |f(z)| = |f(\rho)|$. Write

$$\varphi(z) = (\rho - z) F_p(z) = \rho + \sum_{n=1}^{\infty} (\rho D_n(p) - D_{n-1}(p)) z^n.$$

By Cauchy's inequality, we have

$$\begin{aligned} n|\rho D_n(p) - D_{n-1}(p)| &\le \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\varphi'(re^{i\theta})}{r^{n-1}} \right| d\theta \\ &\le \frac{A_{11}}{r^n} \left(\int_0^{2\pi} |re^{i\theta} - \rho| \left| \frac{f(re^{i\theta})}{r} \right|^p \left| r \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta + \int_0^{2\pi} \left| \frac{f(re^{i\theta})}{r} \right|^p d\theta \right) \\ &\le \frac{A_{11}}{r^n} \left(\int_0^{2\pi} |re^{i\theta} - \rho|^2 \left| \frac{f(re^{i\theta})}{r} \right|^{2p} d\theta \int_0^{2\pi} \left| r \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|^2 d\theta \right)^{1/2} + \frac{A_{12}}{(1-r)^{2p-1}}. \end{aligned}$$

Application of Lemma 1 and Lemma 4 to the above integration gives

$$\begin{aligned} & n|\rho|D_n(p)| - |D_{n-1}(p)|| \\ & \le \frac{A_{13} r^{-n}}{(1-r)^{1/2}} \left(\frac{1}{(1-\rho)^{t(p)}} + \frac{(1-\rho)^{2p}}{(1-r)^{4p-1}} \right)^{1/2} \log^{3/2} \frac{1}{1-r}. \end{aligned} \tag{2.2}$$

Putting $r = 1 - 1/n$, $\rho = 1 - n^{-(4p-1)/(2p+t(p))}$, we have

$$|\rho|D_n(p)| - |D_{n-1}(p)|| \le A_{14} n^{(T(p)-1)/2} \log^{3/2} n. \tag{2.3}$$

Since $|D_n(p)| \le A_{15} n^{2p-1}$, hence

(2.4)

$$\begin{aligned} ||D_n(p)| - |D_{n-1}(p)|| &\le A_{14} n^{(T(p)-1)/2} \log^{3/2} n + (1-\rho)|D_{n-1}(p)| \\ &\le A_{16} \left(n^{(T(p)-1)/2} \log^{3/2} n + n^{2p-(4p^{-1})/(2p+t(p))-1} \right). \end{aligned}$$

Let $x_0 = (4p-1)/(2+t(p))$ $(\frac{1}{4} < p < 1)$. We see that

$$\begin{aligned} 2p - \frac{4p-1}{2p+t(p)} < 2p - x_0 &= \frac{1}{2} + \frac{1}{2} x_0 t(p) \\ &\le \frac{1}{2} + \frac{4p-1}{2p+t(p)} \frac{t(p)}{2} = \frac{1}{2}(1 + T(p)). \end{aligned}$$

Hence the theorem follows from (2.4).

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