ON SUCCESSIVE COEFFICIENTS
OF UNIVALENT FUNCTIONS
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ABSTRACT. Let \( f(z) \in S \), that is, \( f(z) \) is analytic and univalent in the unit
disk \( |z| < 1 \), normalized by \( f(0) = f'(0) - 1 = 0 \). Let \( p \) be real and
\[
\left\{ \frac{f(z)}{z} \right\}^p = 1 + \sum_{n=1}^{\infty} D_n(p)z^n.
\]
Lucas proved that
\[
|D_n(p)| - |D_{n+1}(p)| \leq An^{(t(p)-1)/2} \log^{3/2} n, \quad n = 2, 3, \ldots,
\]
for some absolute constant \( A \) and \( t(p) = (2\sqrt{p}-1)^2 \). In this paper we improve
\( t(p) \) as follows:
\[
T(p) = \frac{4p-1}{2p+t(p)} t(p).
\]

Let the function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S \), and
\[
F_p(z) = \left\{ \frac{f(z)}{z} \right\}^p = 1 + \sum_{n=1}^{\infty} D_n(p)z^n.
\]
It is a very interesting problem to find a best possible number \( t(p) \) for which the
inequality
\[
|D_n(p)| - |D_{n-1}(p)| \leq An^{(t(p)-1)/2}
\]
holds, where \( A \) is an absolute constant.

This problem was first studied by Goluzin. In 1963 Hayman obtained a precise
result \( t(1) = 1 \). In 1956 the author [2] proved that \( t(p) = 2p - 1 \) (\( 0 < p < 1 \)) for
\( f \in S^* \). In the general case, the better result \( t(p) = (2\sqrt{p}-1)^2 \) (\( \frac{1}{4} < p < 1 \)) is due
to Lucas [1].

THEOREM. Let \( f(z) \in S \), and \( F_p(z) = \left\{ \frac{f(z)}{z} \right\}^p = 1 + \sum_{n=1}^{\infty} D_n(p)z^n \). Then
for every \( p \in \left( \frac{1}{4}, 1 \right] \), we have
\[
|D_n(p)| - |D_{n-1}(p)| \leq An^{(T(p)-1)/2} \log^{3/2} n, \quad n = 2, 3, \ldots,
\]
where \( A \) is an absolute constant, and
\[
T(p) = \frac{4p-1}{2p+t(p)} t(p), \quad t(p) = (2\sqrt{p}-1)^2.
\]
This estimate is obviously better than Lucas's because \( (4p-1)/(2p+t(p)) < 1 \)
for \( \frac{1}{4} < p < 1 \). Note that if \( p = \frac{1}{2} \) then \( zF_p(z^2) \) is an odd univalent function, and
so on.
1. Lemmas. We require some lemmas. In the following, $A, A_1, A_2, \ldots$ will denote some absolute constants.

**Lemma 1 [4].** Let $f(z) \in S$. Then

\[ (1.1) \quad \int_0^{2\pi} \left| \frac{r^f(re^{i\theta})}{f(re^{i\theta})} \right|^2 d\theta \leq \frac{A}{1 - r} \log \frac{1}{1 - r}, \quad \frac{1}{2} \leq r < 1. \]

**Lemma 2.** Let $f(z) \in S$ and $F_p(z) = \{f(z)/z\}^p = 1 + \sum_{n=1}^{\infty} D_n(p)z^n$. Let $p$ be given such that $\max_{|z|=\rho} |f(z)| = |f(\rho)|$. Then for every $p \in (\frac{1}{4}, 1]$ we have

\[ (1.2) \quad f \leq \frac{1}{(1 - r)^2} \left( \frac{1}{1 - p} \right)^{1 - (1 - p)^t}, \quad 0 < p < r < 1, \]

\[ \leq \frac{1}{(1 - r)^2} \left( \frac{1}{1 - p} \right)^{1 - (1 - p)^t}, \quad 0 < p \leq r < 1, \]

where $t(p) = (2\sqrt{p} - 1)^2$.

**Proof.** By Goluzin's inequality [5]

\[ |z - \rho| \left| \frac{z - \rho}{z \rho} \right|^2 (1 - \rho^2) - (1 - |z|^2)^2 \]

\[ \leq \left| \frac{1}{f(z)} - \frac{1}{f(\rho)} \right|^2 \left| \frac{f'(z)}{f^2(z)} \right|^2 \left| \frac{\rho f'(\rho)}{f^2(\rho)} \right|^2, \]

where $x_1, x_2$ are real numbers.

The following inequalities are known:

\[ |zf'(z)/f(z)| \leq 1 + \frac{|z|}{1 - |z|}, \quad |z| < 1, \]

\[ (1.4) \quad |zf'(z)/f(z)| \leq 1 + \frac{|z|}{1 - |z|}, \quad |z| < 1, \]

\[ (1.5) \quad t^{-1}(1 - t)^2|f(te^{i\theta})| \leq s^{-1}(1 - s)^2|f(se^{i\theta})|, \quad 0 < s \leq t < 1. \]

From inequality (1.5) and the hypothesis that $0 < \rho \leq r < 1$ and $\max_{|z|=\rho} |f(z)| = |f(\rho)|$, it is easy to show

\[ (1.6) \quad \leq \frac{1}{|f(re^{i\theta})|} + \frac{1}{|f(\rho)|} \leq \frac{1}{|f(re^{i\theta})|} + \frac{1}{|f(\rho e^{i\theta})|} \leq \frac{1}{\rho(1 - r)^2|f(re^{i\theta})|} \leq \frac{2r(1 - p)^2}{\rho(1 - r)^2|f(re^{i\theta})|}. \]

We choose $x_1 = x_2 = \sqrt{p}$ in (1.3) and notice that $|z - \rho| \leq |1 - \rho| \rho$, $|z - \rho| \leq 2r$. Then from (1.4), (1.5), and (1.6) it is not difficult to deduce the first inequality in (1.2).

The second inequality in (1.2) [1] can also be obtained by putting $x_1 = 2\sqrt{p} - 1$, $x_2 = 1$ in (1.3). Thus the proof is complete.

**Lemma 3.** With the above assumption, we have

\[ J(t) = \int_0^{2\pi} \left| t^{e^{i\theta}} - \rho \right|^2 \left| \frac{t^{e^{i\theta}}}{t^{e^{i\theta}}} \right|^2 \left| f(t^{e^{i\theta}}) \right|^{2p} d\theta \]

\[ \leq \begin{cases} \frac{A}{(1 - t)^2(1 - \rho)^{t(p)}} \log \frac{1}{1 - \rho}, & 0 < t \leq \rho < 1, \\ \frac{A(1 - \rho)^{2p}}{(1 - t)^{4p+1}} \log \frac{1}{1 - t}, & 0 < \rho \leq t < 1. \end{cases} \]

**Proof.** The proof follows from Lemmas 1 and 2.
LEMMA 4. With the above assumption, write
\[ \varphi(z) = (\rho - z) \left\{ \frac{f(z)}{z} \right\}^p = \rho + \sum_{n=1}^{\infty} (\rho D_n(p) - D_{n-1}(p)) z^n. \]

Then
\[ I(r) = \int_0^{2\pi} |\varphi(re^{i\theta})|^2 d\theta \]
\[ \leq \begin{cases} \frac{A_1}{(1 - \rho)^2(p)} \log^2 \frac{1}{1 - \rho}, & 1/2 \leq r \leq \rho < 1, \\ \frac{A_2}{(1 - \rho)^2(p)} + \frac{(1 - \rho)^2p}{(1 - r)^4p - 1}, & 1/2 \leq \rho \leq r \leq 1. \end{cases} \]

PROOF. Since
\[ |z\varphi'(z)|^2 = p^2 \left| \frac{z f'(z)}{f(z)} \right|^2 \left( \frac{f(z)}{z} \right)^p - \left( \frac{f(z)}{z} \right)^p \frac{z}{p} \left\{ \frac{f(z)}{z} \right\}^p \]
\[ \leq A_3 \left( |\rho - z|^2 \left| \frac{z f'(z)}{f(z)} \right|^2 \left| \frac{f(z)}{z} \right|^2 + \left| \frac{f(z)}{z} \right|^2 \right) \]
and it is known that
\[ \int_0^{2\pi} \left| \frac{f(te^{i\theta})}{te^{i\theta}} \right|^{4p} d\theta \leq \frac{A_4}{(1 - t)^{4p - 1}} \quad (0 < t < 1), \]
for \( p > \frac{1}{4} \), hence
\[ I'(r) = \frac{d}{dr} \left( \int_0^{2\pi} |\varphi(re^{i\theta})|^2 d\theta \right) = \frac{4}{r} \int_0^r \int_0^{2\pi} |\varphi'(te^{i\theta})|^2 t \, dt \, d\theta \]
\[ = \frac{4}{r} \left( \int_0^{1/2} + \int_0^{r} + \int_0^{r} + \int_0^{r} + \int_0^{r} \right) \left| \varphi'(te^{i\theta}) \right|^2 t \, dt \, d\theta \]
\[ \leq A_5 + A_6 \int_{1/2}^{r} t \, dt \left( \int_0^{2\pi} |te^{i\theta} - \rho|^2 \left| \frac{f'(te^{i\theta})}{f(te^{i\theta})} \right|^2 |f_p(te^{i\theta})|^2 d\theta \right) \]
\[ + \int_{1/2}^{r} A_7 dt \left( \frac{A_8}{(1 - t)(1 - \rho)^{2p}} \log \frac{1}{1 - r}, \quad \frac{1}{2} \leq r < \rho < 1. \right. \]

If \( \frac{1}{2} < r \leq \rho < 1 \), (1.9), together with the first inequality in (1.7), yields
\[ I'(r) \leq \frac{A_8}{(1 - t)(1 - \rho)^{2p}} \log \frac{1}{1 - r}, \quad \frac{1}{2} \leq r < \rho < 1. \]

Integrating both sides of the above inequality with respect to \( r \) from \( \frac{1}{2} \) to \( \rho \) we obtain the first inequality in (1.8).

If \( 1 > r \geq \rho \) we write
\[ \int_0^{2\pi} |te^{i\theta} - \rho|^2 \left| \frac{f'(te^{i\theta})}{f(te^{i\theta})} \right|^2 |f_p(te^{i\theta})|^2 d\theta \]
\[ = \left( \int_0^{1/2} + \int_{1/2}^{r} + \int_r^{r} \right) \int_0^{2\pi} |te^{i\theta} - \rho|^2 \left| \frac{f'(te^{i\theta})}{f(te^{i\theta})} \right|^2 |f_p(te^{i\theta})|^2 d\theta \, dt. \]
By Lemma 3, we have
\[ I'(r) \leq A_9 \left( \frac{1}{(1 - \rho)^t(p) + 1} + \frac{(1 - \rho)^{2p}}{(1 - r)^{4p - 1}} \right) \log \frac{1}{1 - r}, \quad \frac{1}{2} \leq \rho < r < 1. \]
Again integrating both sides of the above inequality from \( \rho \) to \( r \) yields
\[ I(r) \leq I(\rho) + A_{10} \left( \frac{r - \rho}{(1 - \rho)^t(p) + 1} + \frac{(1 - \rho)^{2p}}{(1 - r)^{4p - 1}} \right) \log \frac{1}{1 - r}, \quad \frac{1}{2} \leq \rho \leq r < 1. \]
By the first inequality of (1.8), we get the second inequality in (1.8). Thus the lemma follows.

2. Proof of the Theorem. If \( f \in S \), then the rotation \( e^{-i\varphi}f(e^{i\varphi}z) \) also belongs to \( S \). This rotation does not change the magnitudes of the coefficients of the corresponding function \( F_p(z) \). Thus for a fixed \( \rho \) with \( 0 < \rho < 1 \), there is no loss of generality in supposing that \( \max_{|z|=\rho} |f(z)| = |f(\rho)| \). Write
\[ \varphi(z) = (\rho - z)F_p(z) = \rho + \sum_{n=1}^{\infty} (\rho D_n(p) - D_{n-1}(p))z^n. \]
By Cauchy's inequality, we have
\[ n|\rho D_n(p) - D_{n-1}(p)| \leq \frac{1}{2\pi} \int_{0}^{2\pi} \left| \frac{\varphi'(re^{i\theta})}{re^{i\theta}} \right| d\theta \leq \frac{A_{11}}{r^n} \left( \int_{0}^{2\pi} \left| re^{i\theta} - \rho \right|^{2p} \left| f(re^{i\theta}) \right|^{2p} d\theta \right)^{1/2} + \frac{A_{12}}{(1 - r)^{2p - 1}}. \]
Application of Lemma 1 and Lemma 4 to the above integration gives
\[ n|\rho|D_n(p) - |D_{n-1}(p)| \leq \frac{A_{13}r^{-n}}{(1 - r)^{1/2}} \left( \frac{1}{(1 - \rho)^t(p)} + \frac{(1 - \rho)^{2p}}{(1 - r)^{4p - 1}} \right)^{1/2} \log^{3/2} \frac{1}{1 - r}. \]
Putting \( r = 1 - 1/n, \ \rho = 1 - n^{-(4p-1)/(2p+t(p))} \), we have
\[ |\rho|D_n(p) - |D_{n-1}(p)| \leq A_{14}n^{(T(p)-1)/2} \log^{3/2} n. \]
Since \( |D_n(p)| \leq A_{15}n^{2p-1} \), hence
\[ |D_n(p)| - |D_{n-1}(p)| \leq A_{14}n^{(T(p)-1)/2} \log^{3/2} n + (1 - \rho)|D_{n-1}(p)| \leq A_{16} \left( n^{(T(p)-1)/2} \log^{3/2} n + n^{2p-(4p-1)/(2p+t(p)-1)} \right). \]
Let \( x_0 = (4p - 1)/(2p+t(p))(\frac{1}{2} < p < 1) \). We see that
\[ 2p - \frac{4p - 1}{2p+t(p)} < 2p - x_0 = \frac{1}{2} + \frac{1}{2}x_0t(p) \leq \frac{1}{2} + \frac{4p - 1}{2p+t(p)} \]
Hence the theorem follows from (2.4).
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REFERENCES


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