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SOME EXAMPLES OF CYCLIC VECTORS IN THE DIRICHLET SPACE

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ABSTRACT. We consider the Hilbert space of analytic functions in the open unit disc that have a finite Dirichlet integral. For E, a closed subset of the unit circle with logarithmic capacity zero, we construct a function in this space which is uniformly continuous, vanishes on E, and is cyclic with respect to the shift operator.

For Δ, the open unit disk in the complex plane, let D be the space of functions f analytic on Δ with finite *Dirichlet integral*

$$\iint_{\Delta} |f'|^2 \, dx \, dy < \infty.$$

If f has a Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n,$$

then

$$\|f\|^2 \equiv \sum_{n=0}^{\infty} (n+1) |\hat{f}(n)|^2$$

defines a norm on D which makes D a Hilbert space. The problem of identifying the *cyclic vectors* in D, that is, those functions f such that the set $\mathcal{P} f = \{pf : p \text{ a polynomial}\}$ is dense in D was studied in [**1**]. Denote by $[f]$ the closure of $\mathcal{P} f$. The following result was established.

THEOREM A [**1**, THEOREM 5]. *If f is any function in D whose radial limit vanishes on a set of positive logarithmic capacity then f is not cyclic.*

Recall that if $f \in D$ and T is the unit circle then

$$f^*(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$$

exists for $e^{i\theta} \in T \backslash E$ where E is a set of zero logarithmic capacity depending on f; see [**4**, Chapter IV, Théorème I] for a stronger result.

Question 12 of [**1**] asks if the converse of Theorem A holds for an outer function $f \in D$. That is, letting $Z(f)$ denote the zero set of f^* for an outer function f, if $Z(f^*)$ has zero logarithmic capacity, must f be cyclic? In fact, no examples were given in [**1**] of cyclic vectors f for which $Z(f)$ was uncountable.

In this note, we modify a construction presented by Carleson (see the proof of Theorem 4 in [**2**]) to obtain the following result.

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THEOREM B. *Let E be a closed subset of the circle T of zero logarithmic capacity. Then there exists a function $f \in D$ which is continuous on the closed disk $\overline{\Delta}$, cyclic in D and $Z(f) = E$.*

In the literature there are several definitions of logarithmic capacity. We choose to use the one presented in Chapter III of [**3**]. We prefer the kernel to be positive for $r = |z - w|$, $z, w \in \overline{\Delta}$, and set, following the notation of [**3**],

$$H(t) = \begin{cases} t + \log 4, & t \geq -\log 4, \\ 0, & t < -\log 4, \end{cases}$$

and

$$K(r) = H(\log 1/r) = \log^+(4/r).$$

It is a simple matter to verify that the sets of capacity zero obtained by using the kernel K as in [**3**] are the same as the sets of logarithmic capacity zero obtained by other authors.

If E is a bounded Borel set, let Γ_E be the class of positive measures μ with support contained in E such that $U_\mu(z) \leq 1$, $z \in E$, where U_μ is the *logarithmic potential*

$$U_\mu(z) = \int K(|z - w|)\, d\mu(w).$$

Also, define the *energy integral* $I(\mu)$,

$$I(\mu) = \int U_\mu(z)\, d\mu(z) = \int\int K(|z - w|)\, d\mu(w)\, d\mu(z).$$

The (logarithmic) capacity $C_K(E)$ of E is defined by the relation

$$\operatorname{cap}(E) = C_K(E) = \sup_{\mu \in \Gamma_E} \mu(E) = \sup_{\mu \in \Gamma_E} \|\mu\|.$$

If E is compact there exists a unique measure σ in Γ_E such that $\|\sigma\| = \operatorname{cap}(E)$; see Theorem 6, p. 20 in [**3**]. The potential U_σ is called the *conductor potential* for E.

With this notation and theory established we are ready to begin the proof of Theorem B.

If E is a compact subset of T and $\operatorname{cap}(E) = 0$ we may choose closed subsets I_n of T each consisting of finitely many closed arcs such that

(i) $E \subseteq I_n$, $n = 1, 2, \ldots$,
(ii) $\sum_{n=1}^{\infty} \operatorname{cap}(I_n)^{1/2} < \infty$,
(iii) $\bigcap_{n=1}^{\infty} I_n = E$.

Let U_n be the conductor potential for I_n and σ_n the unique measure defining this potential. We claim that

(1) $U_n = 1$ p.p. on I_n, i.e., $U_n = 1$ on I_n except possibly on a set of capacity zero;
(2) $0 < U_n \leq 1$;
(3) $\|\sigma_n\| = \operatorname{cap}(I_n)$.

These three claims follow easily from [**3**, Theorem 4, p. 19 and Theorem 1, p. 15] and the definition of σ_n. We also claim that

(4) $\hat{\sigma}_n(0)^2 \log^2 4 + \sum_{k=1}^{\infty} |\hat{\sigma}_n(k)|^2/k \leq \operatorname{cap}(I_n)$, where

$$\hat{\sigma}_n(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\theta}\, d\sigma_n(\theta).$$

For this, from (2) and (3) we have

$$I(\sigma_n) \leq \int U_n(z)\, d\sigma_n(z) \leq \sigma_n(I_n) = \|\sigma_n\| = \text{cap}(I_n).$$

By [**4**, Proposition 3, p. 35] we have

$$I(\sigma_n) = \hat{\sigma}_n(0)^2 \log^2 4 + \sum \frac{|\hat{\sigma}_n(k)|^2}{k}.$$

See the proof of [**1**, Theorem 5] for details. This proves (4).

By representing U_n as the Poisson Integral of $U_n(e^{i\theta})$ and using properties (1) and (2) above, we see that for each n we may choose $0 < r_n < 1$ so close to 1 that

$$U_n(r_n e^{i\theta}) \geq \tfrac{1}{2}, \quad \text{for } e^{i\theta} \in E;$$

such an r_n exists since I_n is a union of finitely many arcs. Let U_n^* be the harmonic conjugate of U_n on the disk Δ which vanishes at 0 and set

$$\varphi_n(z) = U_n(r_n z) + iU_n^*(r_n z), \qquad z \in \overline{\Delta}.$$

Since φ_n is analytic on $\overline{\Delta}$ it is clear that $\varphi_n \in D$. We proceed to estimate $\|\varphi_n\|$. First,

$$\hat{\varphi}_n(0) = \sigma_n(T) ln 4 = \hat{\sigma}_n(0) ln 4.$$

Next, for $k > 0$ and $h(t) = ln 4/|1 - e^{it}|$,

$$\begin{aligned} \hat{\varphi}_n(k) &= 2r_n^k \hat{U}_n(k) = 2r_n^k \widehat{h * \sigma_n}(k) \\ &= 2r_n^k \hat{h}(k)\hat{\sigma}_n(k) = \frac{r_n^k \hat{\sigma}_n(k)}{k}. \end{aligned}$$

Thus,

$$\|\varphi_n\|^2 \leq |\hat{\sigma}_n(0)|^2 \log^2 4 + \sum_{k=1}^{\infty} (k+1) \frac{|\hat{\sigma}_n(k)|^2}{k^2} \leq 2\, \text{cap}(I_n),$$

where we have used (4). It follows now that $\sum \varphi_n$ is in D and

$$\left\| \sum \varphi_n \right\| \leq \sqrt{2} \sum \text{cap}(I_n)^{1/2} < \infty.$$

Furthermore, since $0 < \text{Re}\, \varphi_n(z) < 1$ for $z \in \Delta$, we have that

$$f(z) = \exp\left(-\sum \varphi_n(z) \right)$$

is in H^∞; in fact, $\|f\|_\infty \leq 1$.

We claim that

(5) $f \in D$;

(6) $[f] = D$, i.e. f is cyclic in D;

(7) f is continuous on $\overline{\Delta}$;

(8) $Z(f) = E$;

and thus Theorem B is proved.

To prove (5) observe that $f'(z) = f(z) \sum \varphi_n'(z)$. Since $f \in H^\infty$ and $\sum \varphi_n \in D$ it follows easily that $f \in D$.

To verify (6) let $F_N(z) = \exp(-\sum_{n>N} \varphi_n(z))$. Since $\exp(\varphi_n)$ is analytic on a neighborhood of $\overline{\Delta}$ one shows without difficulty that

$$F_N = \prod_{n=1}^{N} \exp(\varphi_n) \cdot f \in [f].$$

It is easily checked that $\lim_{N\to\infty} \|F_N - 1\| = 0$. Thus $1 \in [f]$ and it follows that f is cyclic.

To check (7) and (8) we show first that

$$v(z) = \sum_{n=1}^{\infty} \operatorname{Re}(\varphi_n(z)) = \sum_{n=1}^{\infty} U_n(r_n z)$$

is continuous in the extended sense on $\overline{\Delta}$. If $z_0 \in \overline{\Delta}$ and is a positive distance from E, i.e. $\operatorname{dist}[z_0, E] > 0$, then it follows that for n sufficiently large

$$\operatorname{dist}[r_n z, I_n] \geq \delta > 0$$

for all points z in some open ball centered at z_0. Thus,

$$U_n(z) = \int_{I_n} \log \frac{4}{|r_n z - e^{i\theta}|} \, d\sigma_n(\theta) \leq \left(\log \frac{4}{\delta}\right) \|\sigma_n\| = \left(\log \frac{4}{\delta}\right) \operatorname{cap}(I_n).$$

The Weierstrass M-test implies that v is continuous in a neighborhood of z_0.

Next, for $z_0 \in E$ it follows from the choice of r_n that $v(z_0) = \infty$. Since Re φ_n is continuous at z_0 and Re $\varphi_n(z_0) \geq \frac{1}{2}$, there is a ball $B(z_0, \varepsilon)$ centered at z_0 of radius ε such that $z \in \overline{\Delta} \cap B(z_0, \varepsilon)$ implies Re $\varphi_n(z) \geq \frac{1}{4}$ for $n = 1, 2, \ldots, N$, where N has been chosen beforehand. Thus $v(z) \geq N/4$ for $|z - z_0| < \varepsilon$. This shows that

$$\lim_{z\to z_0} v(z) = \infty, \qquad z \in \overline{\Delta},$$

that is, v is continuous in the extended sense.

The calculations above show that $|f| = e^{-v}$ is continuous on $\overline{\Delta}$ and that $Z(f) = E$. To show f is continuous on $\overline{\Delta}$ we need therefore only check continuity at $e^{it} \in T \backslash E$. Let γ_t be the line segment from 0 to $r_n e^{it}$ and let $\partial/\partial n$ denote the outer normal derivative. Then

$$\begin{aligned} U_n^*(r_n e^{it}) &= \int_{\gamma_t} \frac{\partial U_n(z)}{\partial n} |\, dz| \\ &= \int_{\gamma_t} \left[\frac{\partial}{\partial n} \int_{I_n} \log \frac{4}{|z - e^{i\theta}|} \, d\sigma_n(\theta)\right] |\, dz|, \end{aligned}$$

and

$$|U_n^*(r_n e^{it})| \leq \int_{\gamma_t} \int_{I_n} \frac{c}{|z - e^{i\theta}|} \, d\sigma_n(\theta) |\, dz| \leq \frac{2c\|\sigma_n\|}{\operatorname{dist}[e^{it}, E]}$$

for n sufficiently large. Since $\sum \|\sigma_n\| < \infty$ it follows that $\sum U_n^*$ is continuous at $e^{it} \in T \backslash E$ and the proof is complete.

We wish to make two remarks:

(1) A modification of the above proof of cyclicity can yield that Carleson's original examples (see [**2**, Theorem 4]) are cyclic in D. Of course, these examples may not necessarily be continuous in $\overline{\Delta}$.

(2) If D_α, $0 < \alpha < 1$, is the Hilbert space of analytic functions for which $\|f\|^2 = \sum_{n=0}^{\infty}(n+1)^\alpha|\hat{f}(n)|^2$ is finite, then by using a suitable Bessel capacity instead of logarithmic capacity, one can prove theorems for D_α corresponding to [**1**, Theorem 5] and Theorem B of this note.

REFERENCES

1. L. Brown and A. L. Shields, *Cyclic vectors in the Dirichlet space*, Trans. Amer. Math. Soc. **285** (1984), 269–304.
2. L. Carleson, *Sets of uniqueness for functions regular in the unit circle*, Acta Math. **87** (1952), 325–345.
3. ____, *Selected problems on exceptional sets*, Van Nostrand, Princeton, N.J., 1967.
4. J.-P. Kahane and R. Salem, *Ensembles parfaits et séries trigonométriques*, Actualités Sci. Indust., no. 1301, Hermann, Paris, 1963.

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