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THE UNIQUE REPRESENTATION OF A SELFADJOINT BOUNDED LINEAR FUNCTIONAL

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ABSTRACT. It is well known that every selfadjoint bounded linear functional on a C^*-algebra has a unique minimal decomposition [**6**, Theorem 3.2.5]. In this paper we prove that under some conditions a selfadjoint completely bounded linear map with a unique minimal decomposition is equivalent to the map with a unique commutant representation (up to unitary equivalence). Using the results, we generalize the Gel'fand-Naimark-Segal construction.

1. Introduction. Throughout this paper let M_n denote the C^*-algebra of complex $n \times n$ matrices generated as a linear space by the matrix units E_{ij} $(i, j = 1, 2, \ldots, n)$ and $\mathscr{L}(H)$ the algebra of all bounded linear operators on a Hilbert space H. Let A and B be C^*-algebras and let $L: A \to B$ be a bounded linear map. If for the maps

$$L \otimes I_n: A \otimes M_n \to B \otimes M_n,$$

one has that $\sup_n \|L \otimes I_n\|$ is finite, then L is called *completely bounded* and we let $\|L\|_{\text{cb}}$ denote the sup. The map L is called *positive* provided that $L(a)$ is positive whenever a is positive, and is called *completely positive* if $L \otimes I_n$ is positive for all n. If $L(a^*) = L(a)^*$, we call L *selfadjoint*. Given $S \subseteq \mathscr{L}(H)$, we let S' denote its *commutant*. A selfadjoint completely bounded map ψ has a *minimal decomposition* [**11**, p. 104] provided that ψ can be expressed as a difference of completely positive maps ϕ_1 and ϕ_2 with $\|\psi\|_{\text{cb}} = \|\phi_1\| + \|\phi_2\|$. From [**13**, Proposition 5.1], we know that there are some conditions to ensure the existence of a minimal decomposition but the decomposition is not unique [**11**, p. 107] in general. We also know that the commutant representation for a completely bounded map is not unique [**5**]. However, there are some cases in which we can find that a selfadjoint completely bounded map has a *unique* commutant representation (up to unitary equivalence) if and only if the map has a *unique* minimal decomposition. Applying the above results, we have that each selfadjoint bounded linear functional on a C^*-algebra has a unique commutant representation which generalizes the Gel'fand-Naimark-Segal construction. Finally, the author gratefully acknowledges several valuable conversations with Professor V. Paulsen.

2. The uniqueness problem. Let $\text{CB}(A, \mathscr{L}(H))$ denote the vector space of completely bounded linear maps from C^*-algebra A into $\mathscr{L}(H)$. Let $L \in \text{CB}(A, \mathscr{L}(H))$,

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from [**5**, p. 7] we know that

$$|||L||| = \inf\left\{ \|\phi\| : \begin{bmatrix} \phi & L \\ L^* & \phi \end{bmatrix} \text{ is completely positive} \right\}.$$

We will apply the following theorem [**5**, Theorem 2.10] later:

THEOREM 2.1. *Let* $L \in \mathrm{CB}(A, \mathscr{L}(H))$, *then there exists a Hilbert space* K, *a* $*$*-homomorphism* $\pi\colon A \to \mathscr{L}(K)$, *an isometry* $V\colon H \to K$ *and an operator* $T \in \pi(A)'$ *such that* $[\pi(A)VH]$ *is dense in* K, $\|T\| = |||L|||$ *and* $L = V^*T\pi V$. *If* $L = L^*$, *then* $|||L||| = \|L\|_{\mathrm{cb}}$ *and* $T = T^*$.

We shall refer to a representation of the form given by Theorem 2.1 as a *commutant representation with isometry* of L [**5**, p. 13]. We shall denote it by (π, V, T, K). Suppose that $L \in \mathrm{CB}(A, \mathscr{L}(H))$ and that (π_i, V_i, T_i, K_i), $i = 1, 2$, are two commutant representations of L. We call these representations *unitarily equivalent* provided that there exists a unitary $U\colon K_1 \to K_2$ such that

$$UV_1 = V_2, \quad U\pi_1 = \pi_2 U \quad \text{and} \quad UT_1 = T_2U.$$

From [**5**, Proposition 3.1], we have the following

PROPOSITION 2.2. *Let* $L \in \mathrm{CB}(A, \mathscr{L}(H))$. *Then there is a unique completely positive map* $\phi\colon A \to \mathscr{L}(H)$ *with* $\|\phi\| = |||L|||$ *such that the map*

$$\begin{bmatrix} \phi & L \\ L^* & \phi \end{bmatrix} : A \otimes M_2 \to \mathscr{L}(H) \otimes M_2$$

is completely positive if and only if L has a unique representation (π, V, T, K) (*up to unitary equivalence*).

THEOREM 2.3. *Let* $L = L^* \in \mathrm{CB}(A, \mathscr{L}(H))$ *and* $L(I_A) = kI$, *then the following statements are equivalent*:

(i) *L has a unique minimal decomposition.*

(ii) *There is a unique completely positive map* $\phi\colon A \to \mathscr{L}(H)$ *with* $\|\phi\| = \|L\|_{\mathrm{cb}}$ *such that the map*

$$\begin{bmatrix} \phi & L \\ L & \phi \end{bmatrix} : A \otimes M_2 \to \mathscr{L}(H) \otimes M_2$$

is completely positive.

(iii) *L has a unique representation* (π, V, T, K) (*up to unitary equivalence*) *with* $T = T^*$ *and* $\|T\| = \|L\|_{\mathrm{cb}}$.

PROOF. From Theorem 2.1 and Proposition 2.2, we know that (ii) and (iii) are equivalent. Suppose that (i) is true. By [**5**, Proposition 2.8], there is a completely positive map $\phi\colon A \to \mathscr{L}(H)$ with $\|\phi\| = \|L\|_{\mathrm{cb}}$ such that the map

$$\begin{bmatrix} \phi & L \\ L & \phi \end{bmatrix} : A \otimes M_2 \to \mathscr{L}(H) \otimes M_2$$

is completely positive. It is easy to see that $\frac{1}{2}(\phi + L) - \frac{1}{2}(\phi - L)$ is a minimal decomposition of L. Similarly, if there is another completely positive map $\tilde{\phi}$: $A \to \mathscr{L}(H)$ with $\|\tilde{\phi}\| = \|L\|_{cb}$ such that the map

$$\begin{bmatrix} \tilde{\phi} & L \\ L & \tilde{\phi} \end{bmatrix} : A \otimes M_2 \to \mathscr{L}(H) \otimes M_2$$

is completely positive, then $\frac{1}{2}(\tilde{\phi} + L) - \frac{1}{2}(\tilde{\phi} - L)$ is also a minimal decomposition of L. By (i), we have that $\phi = \tilde{\phi}$. Hence (i) implies (ii). Conversely, if (ii) is true. By [**13**, Proposition 5.1], L has a minimal decomposition $\psi_1 - \psi_2$. Let $\psi = \psi_1 + \psi_2$; by [**10**, Theorem 3.9], we know that

$$\|L\|_{cb} = \min\left\{ \|\phi\|_{cb} \colon \phi \pm L \text{ are completely positive} \right\}.$$

Since $\psi_1 + \psi_2 \pm L$ are completely positive, we have that

$$\|L\|_{cb} = \|\psi_1\| + \|\psi_2\| \leqslant \|\psi_1 + \psi_2\|.$$

Hence $\|\psi\| = \|L\|_{cb}$. Let $\psi = \psi_1 + \psi_2$; applying Stinespring's Theorem [**9**], ψ has a minimal representation $V^*\pi(\cdot)V$. By [**1**, Theorem 1.4.2], ψ_i has representation $V^*T_i\pi(\cdot)V$ with $0 \leqslant T_i \leqslant I$ and $T_i \in \pi(A)'$, $i = 1, 2$. Hence

$$L(\cdot) = V^*(T_1 - T_2)\pi(\cdot)V \quad \text{and} \quad -I \leqslant T_1 - T_2 \leqslant I.$$

Since the matrix

$$\begin{bmatrix} I & T_1 - T_2 \\ T_1 - T_2 & I \end{bmatrix}$$

is positive, by [**5**, Proposition 2.6], the map

$$\begin{bmatrix} \psi & L \\ L & \psi \end{bmatrix} : A \otimes M_2 \to \mathscr{L}(H) \otimes M_2$$

is completely positive. Similarly, if L has another minimal decomposition $\tilde{\psi}_1 - \tilde{\psi}_2$, then the map

$$\begin{bmatrix} \tilde{\psi}_1 + \tilde{\psi}_2 & L \\ L & \tilde{\psi}_1 + \tilde{\psi}_2 \end{bmatrix} : A \otimes M_2 \to \mathscr{L}(H) \otimes M_2$$

is completely positive. By (ii), we have that $\tilde{\psi}_1 + \tilde{\psi}_2 = \psi_1 + \psi_2$ and hence

$$\tilde{\psi}_1 = \tfrac{1}{2}(\tilde{\psi}_1 + \tilde{\psi}_2 + L) = \tfrac{1}{2}(\psi_1 + \psi_2 + L) = \psi_1.$$

Therefore, (ii) implies (i).

REMARK 2.4. Let $L = L^* \in \mathrm{CB}(A, \mathscr{L}(H))$, from the same argument as Theorem 2.3, we know that the map $[\begin{smallmatrix} \phi & L \\ L & \phi \end{smallmatrix}]$ is completely positive if and only if $\phi \pm L$ are completely positive.

EXAMPLE 2.5. Let $\theta(n)$ denote the transpose map of M_n [**13**], then $\theta(n)$ has a unique minimal decomposition [**11**, Theorem 2.2] and $\|\theta(n)\|_{cb} = n$ [**13**]. Now, let ϕ: $M_n \to M_n$ be the completely positive map defined by

$$\phi\big((a_{ij})\big) = \mathrm{Tr}\big((a_{ij})\big)I_n,$$

then $\|\phi\| = \|\theta(n)\|_{cb} = n$. It is not difficult to see that the matrix

$$\begin{bmatrix} \phi(E_{ij}) & \theta(n)(E_{ij}) \\ \theta(n)(E_{ij}) & \phi(E_{ij}) \end{bmatrix}_{ij}$$

is positive. Hence the map

$$\begin{bmatrix} \phi & \theta(n) \\ \theta(n) & \phi \end{bmatrix} : M_n \otimes M_2 \to M_n \otimes M_2$$

defined by

$$\begin{bmatrix} \phi & \theta(n) \\ \theta(n) & \phi \end{bmatrix}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \phi(a) & \theta(n)(b) \\ \theta(n)(c) & \phi(d) \end{bmatrix}$$

is completely positive [**3**, Theorem 2]. By Theorem 2.3, we know that ϕ is unique. Hence $\frac{1}{2}(\phi + \theta(n)) - \frac{1}{2}(\phi - \theta(n))$ is the unique minimal decomposition of $\theta(n)$. Furthermore, let Φ be the Choi's map defined by

$$\Phi\big((a_{ij})\big) = \big\{(n-1)\mathrm{Tr}\big((a_{ij})\big)\big\} I_n - (a_{ij})$$

[**11**, Theorem 1.3], the structure of the segment between Φ and $\theta(n)$ is determined. By Theorem 2.3, each map in the segment has a unique commutant representation.

COROLLARY 2.6. *Let $L = L^*: A \to C$ be a bounded linear map, then L has a unique representation $V^*T\pi V$ (up to unitary equivalence) with $T \in \pi(A)'$, $T = T^*$ and $\|T\| = \|L\|$.*

PROOF. From [**8**, Theorem 2.10], we know that $\|L\| = \|L\|_{\mathrm{cb}}$. By [**6**, Theorem 3.2.5], L has a unique minimal decomposition. By Theorem 2.3, we have proved the corollary.

REMARK 2.7. In Corollary 2.6, for each $a \in A$, we have that $L(a) = \langle \pi(a)TV(I), V(I)\rangle$. It is easy to see that π is a cyclic representation and $V(I)$ is a unit cyclic vector. Hence, we have generalized the Gel'fand-Naimark-Segal construction: To each positive linear functional ϕ, there corresponds uniquely, within unitary equivalence, a representation (π, K) of A with a cyclic vector ξ such that $\phi(a) = \langle \pi(a)\xi, \xi\rangle$.

We give an example of Corollary 2.6 in the following

EXAMPLE 2.8. Let X be a compact Hausdorff space and let $C(X)$ be the C^*-algebra of continuous functions on X with sup norm. Let L be a selfadjoint bounded linear functional on $C(X)$; by the Riesz Theorem [**7**, Theorem 6.19], there corresponds a unique complex regular Borel measure μ such that $L(f) = \int_X f\, d\mu$. By [**7**, Theorem 6.12], there is a Borel function h with $|h| = 1$, such that

$$d\mu = h\, d|\mu| \quad \text{and} \quad \|L\| = \|L\|_{\mathrm{cb}} = |\mu|(X).$$

Since L is selfadjoint, we have that h is real-valued $|\mu|$-a.e. Let $S = \{x \in X: h(x) = 1\}$, then

$$L(f) = \int_S f\, d|\mu| - \int_{X-S} f\, d|\mu|.$$

Let $\phi: C(X) \to C$ be the positive linear map defined by

$$\phi(f) = \int_X f\, d|\mu|.$$

It is easy to see that L has a minimal decomposition $\frac{1}{2}(\phi + L) - \frac{1}{2}(\phi - L)$. By Theorem 2.3 and Remark 2.4, we know that ϕ is the unique positive linear map with $\|\phi\| = |\mu|(X)$ such that the map

$$\begin{bmatrix} \phi & L \\ L & \phi \end{bmatrix} : C(X) \otimes M_2 \to M_2$$

is completely positive. Let V: $C \to L_2(X; |\mu|)$ be the operator defined by $\lambda \to \lambda I$ and π: $C(X) \to \mathcal{L}(L_2(X; |\mu|))$ be the *-representation defined by $\pi(f) = M_f$. It is not difficult to see that

$$\phi(f) = \tilde{V}^* \pi(f) \tilde{V} = \int_X f \, d|\mu|$$

and $[\pi(C(X))\tilde{V}(1)]$ is dense in $L_2(X; |\mu|)$. Let g: $C(X) \to C$ be the map defined by

$$g(x) = \begin{cases} 1 & \text{if } x \in S, \\ -1 & \text{if } x \in X - S, \end{cases}$$

then $M_g \in \pi(C(X))'$, $M_g = M_g^*$, $\|M_g\| = 1$, and

$$\tilde{V}^* M_g \pi(f) \tilde{V} = \int_X gf \, d|\mu| = \int_S f \, d|\mu| - \int_{X-S} f \, d|\mu| = L(f).$$

Hence, L has a unique commutant representation $V^* T \pi V$, where $V = \tilde{V} / \sqrt{|\mu|(X)}$ and $T = |\mu|(X) M_g$.

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