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PARABOLIC B.M.O. AND HARNACK'S INEQUALITY

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ABSTRACT. We present a simplified proof of Moser's parabolic version of the lemma of John and Nirenberg. This lemma is used to prove Harnack's inequality for parabolic equations.

Introduction. In 1964 J. Moser published a proof of a Harnack inequality for nonnegative solutions of a second order parabolic equation in divergence form [**M$_2$**]. The ideas there followed those Moser used in his earlier work on Harnack's inequality in the elliptic case [**M$_1$**]. The most difficult aspect of his proof was the adaptation to the parabolic case of the well-known lemma of F. John and L. Nirenberg [**JN**], which concerns the exponential decay of the distribution function of a function with bounded mean oscillation. Specifically, they proved that if f is defined on a cube C_0 in R^n and, for each subcube C of C_0, there exists a constant a_C for which

$$\frac{1}{|C|}\int_C |f(x) - a_C|\,dx \leqslant A \qquad (|C| = \text{Lebesgue measure of } C)$$

with A independent of C, then

$$\left|\left\{x \in C_0\colon |f(x) - a_{C_0}| > \alpha\right\}\right| \leqslant Be^{-b\alpha/A}|C_0|,$$

where b and B are constants depending only on dimension.

The difficulty in adapting the above result to the parabolic case is the special role played by the time variable. In fact, Harnack's inequality for a nonnegative solution of a parabolic equation is a control or bound of the value of such a function at a given time in terms of its value at a later time. This necessary time lag had to be reflected in a new John-Nirenberg lemma which Moser did formulate in [**M$_2$**], but the proof of which was difficult to follow. Moser himself published another proof of Harnack's inequality for the parabolic case in 1971 [**M$_2$**] with the expressed purpose of avoiding the version of the John-Nirenberg lemma he had previously formulated.

The purpose of this note is to return to Moser's original method for establishing Harnack's inequality in the parabolic setting and to present a simplified proof of the parabolic John-Nirenberg lemma (called Main Lemma in [**M$_2$**]). Our simplification

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is based on A. P. Calderón's proof of the original John-Nirenberg lemma [**N**]. We mention that in the elliptic case results related to ours are contained in a paper by R. Hanks [**H**].

In what follows, x and y denote points in R^n, while s and t stand for real numbers. We set

$$U = \{(x,t)\colon |x_i| < 1, i = 1,\dots,n, |t| < 1\},$$
$$U^+ = \{(x,t) \in U\colon 0 < t < 1\}, \quad U^- = \{(x,t) \in U\colon -1 < t < 0\},$$
$$V^+ = \{(x,t) \in U\colon 1/2 < t < 1\}, \quad V^- = \{(x,t) \in U\colon -1 < t < -1/2\}.$$

A parabolic rectangle C of R^{n+1} is the image of U through a transformation of the form

$$\pi(x,t) \equiv \pi_{(\alpha,\gamma)}^{(x_0,t_0)}(x,t) \equiv \left(x_0 + \alpha x, t_0 + (\gamma\alpha)^2 t\right).$$

Here $x_0 \in R^n$, $t_0 \in R$, $\alpha > 0$, and $\gamma > 0$ are fixed. C^+, C^-, D^+ and D^- will represent those subrectangles of C which are the images through π of U^+, U^-, V^+, and V^-, respectively. Note that C^+, C^-, D^+ and D^- are themselves parabolic subrectangles of R^{n+1}. Constants which depend only on α and γ will be called dimensional constants. Finally, for our results we must restrict the family of parabolic rectangles to those corresponding to a $\gamma \geqslant \gamma_0 > 0$ with γ_0 fixed. (2^{-100} will do.)

Parabolic B.M.O. with a time lag. Our starting point will be Moser's result concerning the logarithm of a positive solution $u(x,t)$ to the equation

$$\frac{\partial u}{\partial t}(x,t) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}(x,t)\right),$$

where the matrix $A(x,t) = (a_{ij}(x,t))$ is bounded and positive definite uniformly in the variables (x,t). Moser's result, which is derived from the equation, can be stated as follows [$\mathbf{M_2}$, pp. 119–124]: "Given $f = \log(1/u)$ there exist a number $A > 0$ and, for each parabolic subrectangle C of U, a number a_C such that

$$(1) \qquad \begin{aligned} &\frac{1}{|C^+|}\int_{C^+} \sqrt{(f(x,t) - a_C)^+}\,dx\,dt \leqslant A, \\ &\frac{1}{|C^-|}\int_{C^-} \sqrt{(a_C - f(x,t))^+}\,dx\,dt \leqslant A," \end{aligned}$$

where a^+ denotes the positive part of a.

Our aim is to prove

THEOREM 1 (PARABOLIC JOHN AND NIRENBERG). *If f satisfies condition* (1) *for each parabolic subrectangle C of a fixed one C_0, then there exist two dimensional constants B and b such that for every $\alpha > 0$,*

$$(2) \qquad \begin{aligned} &\left|\left\{(x,t) \in D_0^+ : \left(f(x,t) - a_{C_0}\right)^+ > \alpha\right\}\right| \leqslant Be^{-b\alpha/A}|D_0^+|, \\ &\left|\left\{(x,t) \in D_0^- : \left(a_{C_0} - f(x,t)\right)^+ > \alpha\right\}\right| \leqslant Be^{-b\alpha/A}|D_0^-|. \end{aligned}$$

We begin the proof of Theorem 1 by first proving a weaker exponential decay, namely

THEOREM 2. *Assume f satisfies condition* (1) *for each parabolic subrectangle C of a fixed one C_0. Then there exist two positive dimensional constants B and b such that for every $\alpha > 0$,*

$$\left|\left\{(x,t) \in D_0^+ : \left(f(x,t) - a_{C_0}\right)^+ > \alpha\right\}\right| \leqslant Be^{-b(\alpha/A)^{1/2}}|D_0^+|, \tag{3}$$
$$\left|\left\{(x,t) \in D_0^- : \left(a_{C_0} - f(x,t)\right)^+ > \alpha\right\}\right| \leqslant Be^{-b(\alpha/A)^{1/2}}|D_0^-|.$$

PROOF. By translation we may assume the origin is the center of C_0. Note that it is sufficient to prove only the first of the two inequalities in (3), since the second is a consequence of the first applied to the function $-f(x,-t)$. Finally, without loss of generality, we may assume $A = 1$ and $a_{C_0} = 0$. We then want to show

$$\left|\left\{(x,t) \in D_0^+ : f(x,t)^+ > \alpha\right\}\right| \leqslant Be^{-b\alpha^{1/2}}|D_0^+| \tag{4}$$

for suitable dimensional constants B and b.

Selection process. We parabolically subdivide D_0^+ into 4^{n+2} congruent parabolic subrectangles by dividing each spatial side of D_0^+ into four equal parts and the time interval in sixteen equal parts. For each such subrectangle D_1^+ we associate the corresponding rectangle C_1, and we set aside (or select) D_1^+ if $\alpha < a_{C_1}^+$. If $a_{C_1}^+ \leqslant \alpha$ we subdivide D_1^+ in the same manner as above and set aside those parabolic subrectangles D_2^+ for which $\alpha < a_{C_2}^+$. We continue this process and collect those rectangles $\{D_\nu^+(\alpha)\}$, $\nu = 1, 2, \ldots$, which have been set aside.

Set $D(\alpha) = \bigcup_\nu D_\nu^+(\alpha)$. We will show

$$|D(\alpha)| \leqslant Be^{-b\alpha^{1/2}}|D_0^+| \tag{5}$$

for suitable B and b depending only on dimension. This implies estimate (4), since we

Claim. For almost every $(x,t) \in D_0^+ \setminus D(\alpha)$,

$$\sqrt{f(x,t)^+} \leqslant 1 + \sqrt{\alpha}.$$

In fact, if $(x,t) \in D_0^+ \setminus D(\alpha)$, then there exists a sequence of parabolic rectangles D_j^+ such that $(x,t) \in D_j^+$, $a_{C_j} \leqslant \alpha$, and $|D_j^+| \to 0$. Hence,

$$\frac{1}{|C_j^+|}\int_{C_j^+} \sqrt{f^+} \leqslant \frac{1}{|C_j^+|}\int_{C_j^+} \sqrt{\left(f - a_{C_j}\right)^+} + \sqrt{\alpha} \leqslant 1 + \sqrt{\alpha}.$$

(Remember $A = 1$.) For almost every $(x,t) \in D_0^+ \setminus D(\alpha)$,

$$\frac{1}{|C_j^+|}\int_{C_j^+} \sqrt{f^+} \to \sqrt{f(x,t)^+}.$$

(See **[JMZ]**.) This proves our Claim. To prove estimate (5) for $D(\alpha)$ it is sufficient to prove that if $\beta^{1/2} > \alpha^{1/2} + 1$ then

$$(6) \qquad |D(\beta)| \leqslant \left(C/(\beta^{1/2} - \alpha^{1/2} - 1)\right)|D(\alpha)|,$$

with C depending only on the dimension. In fact, if (6) holds then the function $g(\alpha) = |D(\alpha^2)|$ satisfies $g(\beta) \leqslant (C/(\beta - \alpha - 1))g(\alpha)$, provided $\beta > \alpha + 1$. Hence, if $L = 2C + 1$ then $g(\alpha + L) \leqslant \frac{1}{2}g(\alpha)$ for $\alpha > 0$, so

$$|D(\alpha^2)| = |g(\alpha)| \leqslant Be^{-b\alpha}g(1) \leqslant Be^{-b\alpha}|D_0^+|.$$

Our final aim is to prove (6). We return momentarily to the family $D(\alpha)$.

For each $D_\nu^+(\alpha)$ there is the corresponding $C_\nu^-(\alpha)$. While the family $\{D_\nu^+(\alpha)\}$ obviously does not overlap, the family $\{C_\nu^-(\alpha)\}$ certainly may. For technical reasons, which will soon be clear, we have to select a "maximal" nonoverlapping subfamily of $\{C_\nu^-(\alpha)\}$, and we do this in the following manner. At the first step of the selection process, pick a nonoverlapping subfamily $\{\overset{*}{C}{}_1^-(\alpha)\}$ of the finite family $\{C_1^-(\alpha)\}$, with the property that each $C_1^-(\alpha)$ overlaps with some $\overset{*}{C}{}_1^-(\alpha)$. At the second step of the selection process, pick a nonoverlapping subfamily $\{\overset{*}{C}{}_2^-(\alpha)\}$ of the finite family $\{C_2^-(\alpha)\}$, whose members do not overlap with any $\overset{*}{C}{}_1^-(\alpha)$ and for which each $C_2^-(\alpha)$ overlaps with some $\overset{*}{C}{}_1^-(\alpha)$ or $\overset{*}{C}{}_2^-(\alpha)$. In general, at the νth step of the selection process we pick a nonoverlapping subfamily $\{\overset{*}{C}{}_\nu^-(\alpha)\}$ of the finite family $\{C_\nu^-(\alpha)\}$, whose members do not overlap with any previous $\overset{*}{C}{}_\lambda^-(\alpha)$, $\lambda < \nu$, and for which any $C_\nu^-(\alpha)$ overlaps with some $\overset{*}{C}{}_\lambda^-(\alpha)$, $\lambda \leqslant \nu$.

Now pick $0 < \alpha < \beta$ and let $\{D_\nu^+(\alpha)\}$ and $\{D_\nu^+(\beta)\}$ denote the parabolic rectangles chosen in the selection process for the numbers α and β, respectively. $\{\overset{*}{C}{}_\nu^-(\beta)\}$ denotes the "maximal" subfamily of $\{C_\nu^-(\beta)\}$ described above. Recall, $D(\beta) = \bigcup_\nu D_\nu^+(\beta)$ and $D(\alpha) = \bigcup_\nu D_\nu^+(\alpha)$.

$$|D(\beta)| = \sum_\eta |D_\eta^+(\beta)| = \sum_\mu \left(\sum_{\eta \in I_\mu} |D_\eta^+(\beta)| \right),$$

where $I_\mu = \{\eta\colon C_\eta^-(\beta)$ overlaps $\overset{*}{C}{}_\mu^-(\beta)$ and does not overlap any $\overset{*}{C}{}_\lambda^-(\beta)$ for $\lambda < \mu$, i.e., does not overlap any $\overset{*}{C}{}_\lambda^-(\beta)$ chosen prior to the selection step when $\overset{*}{C}{}_\mu^-(\beta)$ was taken$\}$. For $\eta \in I_\mu$ the length of the sides of $C_\eta^-(\beta)$ is less than or equal to the length of the corresponding sides of $\overset{*}{C}{}_\mu^-(\beta)$. Hence there exists a dimensional constant N such that $D_\eta^+(\beta) \subset N\overset{*}{C}{}_\mu^-(\beta)$ for each $\eta \in I_\mu$, where $N\overset{*}{C}{}_\mu^-(\beta)$ is obtained from $\overset{*}{C}{}_\mu^-(\beta)$ by expanding each side, symetrically about the center, to a length of N times the original length. Therefore, $\bigcup_{n \in I_\mu} D_\eta^+(\beta) \subset N\overset{*}{C}{}_\mu^-(\beta)$, so

$$\sum_{\eta \in I_\mu} |D_\eta^+(\beta)| = \left| \bigcup_{\eta \in I_\mu} D_\eta^+(\beta) \right| \leqslant C\left|\overset{*}{C}{}_\mu^-(\beta)\right|.$$

We conclude that $|D(\beta)| \leqslant C\sum_\mu |\overset{*}{C}{}_\mu^-(\beta)|$.

Remember that each $\overset{*}{C}{}_\mu^-(\beta)$ is associated with a $D_\mu^+(\beta)$ chosen in the selection process. It is clear from this process that for $\beta > \alpha$, $D_\mu^+(\beta) \subset D_\nu^+(\alpha)$ for some ν. Since the family $D_\nu^+(\alpha)$ is pairwise nonoverlapping, we have

$$(7) \qquad |D(\beta)| \leqslant C\sum_\nu \sum_{\mu \in J_\nu} \left|\overset{*}{C}{}_\mu^-(\beta)\right|,$$

where $J_\nu = \{\mu\colon D_\mu^+(\beta) \subset D_\nu^+(\alpha)\}$.

Now,

$$\beta^{1/2} < \sqrt{a^{+}_{\overset{*}{C}_\mu(\beta)}} \leqslant \frac{1}{|\overset{*}{C}{}^{-}_{\mu}(\beta)|} \int_{\overset{*}{C}{}^{-}_{\mu}(\beta)} \sqrt{\left(a_{\overset{*}{C}_\mu(\beta)} - f\right)^{+}} + \frac{1}{|\overset{*}{C}{}^{-}_{\mu}(\beta)|} \int_{\overset{*}{C}{}^{-}_{\mu}(\beta)} \sqrt{f^{+}}\,.$$

Recall the normalization $A = 1$ in property (1) of f. Using this and the fact that the family $\{\overset{*}{C}_\mu(\beta)\}$ is pairwise nonoverlapping, we have

$$(\beta^{1/2} - 1) \sum_{\mu \in J_\nu} \left|\overset{*}{C}{}^{-}_{\mu}(\beta)\right| \leqslant \int_{\bigcup_{\mu\in J_\nu} \overset{*}{C}{}^{-}_{\mu}(\beta)} \sqrt{f^{+}}\,.$$

For $\mu \in J_\nu$, $D^{+}_{\mu}(\beta) \subset D^{+}_{\nu}(\alpha)$, and $D^{+}_{\nu}(\alpha)$ was obtained by subdividing a previous parabolic rectangle $D^{+}_{\nu\_}(\alpha)$, for which $a^{+}_{C_{\nu\_}(\alpha)} \leqslant \alpha$. Our method of subdivision guarantees $\overset{*}{C}{}^{-}_{\mu}(\beta) \subset C^{+}_{\nu\_}(\alpha)$ for each $\mu \in J_\nu$. Hence,

$$\begin{aligned}\int_{\bigcup_{\mu\in J_\nu} \overset{*}{C}{}^{-}_{\mu}(\beta)} \sqrt{f^{+}} &\leqslant \int_{\bigcup_{\mu\in J_\nu} \overset{*}{C}{}^{-}_{\mu}(\beta)} \sqrt{\left(f - a_{C_{\nu\_}(\alpha)}\right)^{+}} + \alpha^{1/2} \sum_{\mu\in J_\nu} \left|\overset{*}{C}{}^{-}_{\mu}(\beta)\right| \\ &\leqslant \int_{C^{+}_{\nu\_}(\alpha)} \sqrt{\left(f - a_{C_{\nu\_}(\alpha)}\right)^{+}} + \alpha^{1/2} \sum_{\mu\in J_\nu} \left|\overset{*}{C}{}^{-}_{\mu}(\beta)\right|.\end{aligned}$$

Finally, we have

$$(\beta^{1/2} - \alpha^{1/2} - 1) \sum_{\mu\in J_\nu} \left|\overset{*}{C}{}^{-}_{\mu}(\beta)\right| \leqslant C|D^{+}_{\nu}(\alpha)|,$$

and if we substitute this into (7) we obtain

$$|D(\beta)| \leqslant \left(C/(\beta^{1/2} - \alpha^{1/2} - 1)\right)|D(\alpha)|,$$

which was to be proved.

As an immediate corollary we have

THEOREM 3. *If f satisfies* (1) *for each parabolic subrectangle C of $U = \{(x,t)$: $|x| \leqslant 1, |t| \leqslant 1\}$, then there exists $A > 0$ such that for every such C and corresponding D^{+} and D^{-},*

$$\begin{aligned}&\frac{1}{|D^{+}|}\int_{D^{+}} \left(f(x,t) - a_C\right)^{+} dx\,dt \leqslant A,\\ (8)\qquad &\frac{1}{|D^{-}|}\int_{D^{-}} \left(a_C - f(x,t)\right)^{+} dx\,dt \leqslant A.\end{aligned}$$

With Theorem 3 established, an almost verbatim repetition of the argument used to show Theorem 2 can now be used to prove

THEOREM 4. *Assume f satisfies condition* (8) *for each parabolic subrectangle C of a fixed one $C_0 = \{(x,t): |x - x_0| \leqslant r, |t - t_0| \leqslant r^2\}$. Set $D^{+}_{0} = C_0 \cap \{(x,t): t \geqslant t_0 + \frac{1}{2}r^2\}$ and $D^{-} = C_0 \cap \{(x,t): t \leqslant t_0 - \frac{1}{2}r^2\}$. There exist two positive dimensional constants B and b such that for every $\alpha > 0$,*

$$\left|\left\{(x,t) \in D^{+}_{0} : \left(f(x,t) - a_{C_0}\right)^{+} > \alpha\right\}\right| \leqslant Be^{-b\alpha/A}|D^{+}_{0}|$$

and

$$\left|\left\{(x, t) \in D_0^-: \left(a_{C_0} - f(x, t)\right)^+ > \alpha\right\}\right| \leqslant Be^{-b\alpha/A}|D_0^-|.$$

It is now easy to prove Theorem 4 in [$\mathbf{M}_2$]. We state it as

COROLLARY 5. *Let $u(x, t)$ be a positive solution to*

$$\frac{\partial u}{\partial t}(x, t) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x, t)\right)\frac{\partial u}{\partial x_j}(x, t) \quad in \left\{(x, t): |x| < 2, |t| < 1\right\}.$$

The matrix $(a_{ij}(x, t))$ is assumed to be real, symmetric, bounded, and uniformly positive definite, i.e., $\exists\lambda > 0$ such that

$$\lambda|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \leqslant \lambda^{-1}|\xi|^2$$

for each $(x, t) \in \mathbf{R}^{n+1}$ and each $\xi \in \mathbf{R}^n$. Let $D^+ = \{(x, t): |x| < 1, \frac{1}{2} < t < 1\}$ and $D^- = \{(x, t): |x| < 1, -1 < t < -\frac{1}{2}\}$. Then there exist constants $\delta > 0$ and $C > 0$, depending only on λ and n, such that

$$\left(\int_{D^-} u(x, t)^{\delta}\, dx\, dt\right)\left(\int_{D^+} u(x, t)^{-\delta}\, dx\, dt\right) \leqslant C.$$

PROOF. As was pointed out in the Introduction, Moser proved that $f = \log(1/u)$ satisfies condition (1) in $C_0 = \{(x, t): |x| < 1, |t| < 1\}$ with A depending only on λ and n. Hence the parabolic John-Nirenberg theorem stated at the beginning of this section is valid. In particular, if $\delta < b/A$ then

$$\begin{aligned}\int_{D_0^-} u(x, t)^{\delta}\, dx\, dt &= \int_{D_0^-} e^{-\delta f(x,t)}\, dx\, dt \\ &\leqslant e^{-\delta a_{C_0}} \int_{D_0^-} e^{\delta(a_{C_0} - f(x,t))^+}\, dx\, dt \leqslant Be^{-\delta a_{C_0}}\delta \int_0^{\infty} e^{\delta\alpha} e^{-b/A\alpha}\, d\alpha.\end{aligned}$$

In the same way we obtain

$$\begin{aligned}\int_{D_0^+} u(x, t)^{-\delta}\, dx\, dt &\leqslant e^{\delta a_{C_0}} \int_{D_0^+} e^{\delta(f(x,t) - a_{C_0})^+}\, dx\, dt \\ &\leqslant Be^{\delta a_{C_0}}\delta \int_0^{\infty} e^{(\delta - b/A)\alpha}\, d\alpha.\end{aligned}$$

The conclusion is now clear.

We end this note with the remark that Theorem 3 and, hence, also Theorem 4 do not change if the power $\frac{1}{2}$ in condition (1) is replaced by a power p with $0 < p < 1$. The power $\frac{1}{2}$ in (3) is replaced by p. The proof of Theorem 2 does not change in any essential way.

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