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MINIMAL SUPERALGEBRAS OF WEAK-* DIRICHLET ALGEBRAS

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ABSTRACT. Let A be a weak-* Dirichlet algebra in $L^\infty(m)$ and let $H^\infty(m)$ be the weak-* closure of A in $L^\infty(m)$. It may happen that there are minimal weak-* closed subalgebras of $L^\infty(m)$ that contain $H^\infty(m)$ properly. In this paper it is shown that if there is a minimal, proper, weak-* closed superalgebra of $H^\infty(m)$, then, in fact, that algebra is the unique least element in the lattice of all proper weak-* closed superalgebras of $H^\infty(m)$.

Recall that by definition [6], a weak-* Dirichlet algebra is an algebra A of essentially bounded measurable functions on a probability measure space $(X, \mathscr{A}, m)$ such that (i) the constant functions lie in A; (ii) $A + \bar{A}$ is weak-* dense in $L^\infty = L^\infty(m)$ (the bar denotes conjugation); and (iii) for all f and g in A,

$$\int_X fg\,dm = \left(\int_X f\,dm\right)\left(\int_X g\,dm\right).$$

The abstract Hardy spaces $H^p = H^p(m)$, $1 \le p \le \infty$, associated with A are defined as follows. For $1 \le p < \infty$, H^p is the closure of A in L^p, while H^∞ is defined to be the weak-* closure of A in L^∞. The space H^∞ is a weak-* closed subalgebra of L^∞.

In recent years the structure of the lattice $\mathscr{L}$ of proper weak-* closed superalgebras of H^∞ has attracted considerable attention; see, in particular, [2, 3 and 5]. It is easy to construct examples where $\mathscr{L}$ has no least element and no minimal elements. However, in Corollary 5 of [5], we gave a necessary and sufficient condition for $\mathscr{L}$ to have a least element and we characterized it in Corollary 3 of [3]. The question arises: Can $\mathscr{L}$ have minimal elements, but no least element? In this paper we show that the answer is no.

THEOREM. *If the lattice $\mathscr{L}$ of proper weak-* closed superalgebras of H^∞ has a minimal element, then that element is the least element of $\mathscr{L}$.*

Let B be a weak-* closed superalgebra of A. Let $B_0 = \{f \in B; \int_X f\,dm = 0\}$ and let I_B be the largest weak-* closed ideal of B contained in B_0. Then by Lemma 2 of [4], $I_B = \{f \in L^\infty; \int_X fg\,dm = 0 \text{ for all } g \in B\}$. If $B = H^\infty$, then $I_B = H_0^\infty$. On p. 153 of [4], the measure m is called quasi-multiplicative on B if $\int_X f^2\,dm = 0$ for every

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$f \in B$ such that $\int_E f\,dm = 0$ for all sets E such that the characteristic function $\chi_E \in B$. The measure m is clearly quasi-multiplicative on H^∞ and L^∞. More elaborate examples are given on p. 163 of [**4**]. However, recently Kallenborn and König [**1**, Theorem 1.5] showed that m is always quasi-multiplicative on any weak-* closed superalgebra of A. This fact will play a crucial role in the proof of the theorem.

For any subset $M \subseteq L^\infty$, $[M]_2$ will denote the closed linear span of M in L^2. If E is a measurable subset of X, χ_E will denote the characteristic function of E. The support set of any function $f \in L^1$ will be denoted $E(f)$.

LEMMA 1. *Let B be a weak-* closed superalgebra of A. If $f \in B$ and $\chi_E f \notin I_B$ for every $\chi_E \in B$ with $\chi_E f \neq 0$, then $\chi_{E(f)} \in B$.*

PROOF. Set $M_f = [fB]_2$. Then M_f is a left-continuous invariant subspace for B; i.e., $\chi_E M_f \supsetneqq \chi_E[I_B M_f]_2$ for every nonzero $\chi_E \in B$ with $\chi_E M_f \neq \{0\}$. Since the measure m is quasi-multiplicative on B, by Theorem 1.5 of [**1**], we may apply Theorem 2 of [**4**] to conclude that $M_f = \chi_{E_0} q[B]_2$ for some unimodular q and some $\chi_{E_0} \in B$. Clearly $\chi_{E(f)} = \chi_{E_0}$, and so $\chi_{E(f)} \in B$.

LEMMA 2. *Suppose that B is a minimal weak-* closed superalgebra of H^∞. If $f \in H^2$ and $f \notin [I_B]_2$, then $|f| > 0$ a.e.*

PROOF. Let K denote the orthogonal complement of I_B in H^2. We first show that if $f \in K, f \neq 0$, then $|f| > 0$ a.e. To see this, set $g = hf$ where $h \in H^\infty$, $[hA]_2 = H^2$, and $|h| = \min\{1, 1/|f|\}$. Then $g \in B$, and $\chi_E g \notin I_B$ for every $\chi_E \in B$ with $\chi_E g \neq 0$. Lemma 1 implies that $\chi_{E(f)} = \chi_{E(g)}$ belongs to B. Set $M_f = [fA]_2$ and $D = \{g \in B;\ gM_f \subseteq M_f\}$. Then D is a weak-* closed superalgebra of H^∞ with $D \subseteq B$, and $\chi_{E(f)} \in D$. If $\chi_{E(f)} \neq 1$, then $H^\infty \subsetneqq D$, and so $D = B$ since B is assumed to be minimal. But then, since $M_f \subseteq H_0^2 = \{g \in H^2;\ \int_X g\,dm = 0\}$ and $BM_f \subseteq M_f$, we see that $M_f \subseteq [I_B]_2$ by Lemma 2 of [**4**]. Thus we conclude that $f \in [I_B]_2$, contrary to our hypothesis that $f \in K$. Thus $\chi_{E(f)} = 1$. To complete the proof, choose an arbitrary $f \in H^2$ with $f \notin [I_B]_2$ and write f as $f = u + f_0$ where $u \in K$ and $f_0 \in [I_B]_2$. Since $f \notin [I_B]_2$, $u \neq 0$, and so $|u| > 0$ a.e., by what we just proved. Again, set $g = hf$, where $h \in H^\infty$, $[hA]_2 = H^2$, and $|h| = \min\{1, 1/|f|\}$. Then $g \in B$ and we claim that $\chi_E g \notin I_B$ for every $\chi_E \in B$ with $\chi_E g \neq 0$. For, if $\chi_{E_0} g \in I_B$, for some $\chi_{E_0} \in B$ with $\chi_{E_0} g \neq 0$, then $\chi_{E_0} hu \in I_B$. Since the equation $[hA]_2 = H^2$ implies that $[hI_B]_2 = [I_B]_2$, we find that $\chi_{E_0} u \in I_B$, which contradicts the fact that $|u| > 0$ a.e. Lemma 1 now implies that $\chi_{E(f)} = \chi_{E(g)}$ lies in B. So $(1 - \chi_{E(f)})u = -(1 - \chi_{E(f)})f_0$ belongs to $[I_B]_2 \cap K = \{0\}$. Thus $(1 - \chi_{E(f)})u = 0$ a.e., which implies that $\chi_{E(f)} = 1$ a.e., since $|u| > 0$ a.e.

PROOF OF THE THEOREM. Let B be a minimal, proper, weak-* closed superalgebra of A, and let D be any proper weak-* closed superalgebra of A. We must show that $B \subseteq D$. By Lemma 2 of [**4**], it suffices to show that $I_D \subseteq I_B$. Since $D \supsetneqq H^\infty$, there is a $\chi_E \in D$ with $0 \lneqq m(E) \lneqq 1$, by Lemma 3 of [**3**]. If $f \in I_D$, then both $\chi_E f$ and

$(1-\chi_E)f \in I_D$ and, in particular, $\chi_E f$ and $(1-\chi_E)f$ belong to H^∞. By Lemma 2, both $\chi_E f$ and $(1-\chi_E)f$ belong to I_B, and so $f = \chi_E f + (1-\chi_E)f$ belongs to I_B. Thus $I_D \subseteq I_B$ and this completes the proof.

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References

1. R. Kallenborn and H. König, *An invariant subspace theorem in the abstract Hardy algebra theory*, Arch. Math. **39** (1982), 51–58.

2. P. S. Muhly, *Maximal weak-* Dirichlet algebras*, Proc. Amer. Math. Soc. **36** (1972), 515–518.

3. T. Nakazi, *Superalgebras of weak-* Dirichlet algebras*, Pacific J. Math. **68** (1977), 197–207.

4. ______, *Invariant subspaces of weak-* Dirichlet algebras*, Pacific J. Math. **69** (1977), 151–167.

5. ______, *Quasi-maximal ideals and quasi-primary ideals of weak-* Dirichlet algebras*, J. Math. Soc. Japan **31** (1979), 677–685.

6. T. P. Srinivasan and J.-K. Wang, *Weak-* Dirichlet algebras*, Function Algebras, Scott, Foresman, and Co., Chicago, 1966, pp. 216–249.

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