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ON TWO CONJECTURES CONCERNING THE PARTIAL SUMS OF THE HARMONIC SERIES

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ABSTRACT. Let S_n denote the nth partial sum of the harmonic series. For a given positive integer $k > 1$, there exists a unique integer n_k such that $S_{n_k - 1} < k < S_{n_k}$. It has been conjectured that n_k is equal to the integer nearest $e^{k-\gamma}$, where γ is Euler's constant. We provide an estimate on n_k which suggests that this conjecture may have to be modified. We also propose a conjecture concerning the amount by which $S_{n_k - 1}$ and S_{n_k} differ from k.

1. The first conjecture. Let

$$S_n = \sum_{m=1}^{n} \frac{1}{m} \tag{1}$$

denote the nth partial sum of the harmonic series. It is a well-known fact (see [**7**, pp. 380–381]) that S_n is never an integer for $n > 1$. Since the harmonic series diverges, there exists for each integer $k > 1$ a corresponding integer n_k such that

$$S_{n_k - 1} < k < S_{n_k}. \tag{2}$$

The question we wish to address here is: What is the value of n_k for each $k > 1$? It has been conjectured in *Advanced Problems* 5346 (1967, p. 209) and 5989 (1974, p. 910) in the American Mathematical Monthly that n_k is equal to $((e^{k-\gamma}))$, where $\gamma = 0.5772156649 \cdots$ is the Euler-Mascheroni constant and $((x))$ denotes the integer nearest x. L. Comtet [**6**, p. 209] shows that either $n_k = [e^{k-\gamma}]$ or $[e^{k-\gamma}] + 1$, where $[x]$ denotes the greatest integer $\leq x$. R. P. Boas comments [**2**, p. 749] in a partial solution to this problem that this conjecture has been verified by computation for $k \leq 200$ by Boas (unpublished) and for $k \leq 1000$ by R. Spira (unpublished). Boas also establishes [**5**, p. 865] the following partial result: If m and δ are the integral and fractional parts of $e^{k-\gamma}$, then $n_k = ((e^{k-\gamma}))$ provided that $\delta \notin [1/2 - 1/10m, 1/2 + 1/m]$.

The following result suggests that the above conjecture may possibly be improved.

THEOREM. *For each integer $k > 1$, the integer n_k satisfies the inequality*

$$-\frac{2}{e^{3(k-\gamma)}} - \frac{1}{2} < n_k - \left(e^{k-\gamma} - \frac{1}{24e^{k-\gamma}}\right) < +\frac{1}{2} + \frac{2}{e^{3(k-\gamma)}}. \tag{3}$$

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REMARK. This theorem suggests that n_k may be equal to the integer nearest $e^{k-\gamma} - 1/(24e^{k-\gamma})$. On one hand, the original conjecture could perhaps be modified to reflect the new information contained in the above theorem. On the other hand, this theorem also suggests that we are not yet in a position to postulate a "correct" conjecture as to the value of n_k.

Before proving this theorem, we digress in order to pose the following question: Is it true that

$$((e^{k-\gamma})) = \left(\left(e^{k-\gamma} - \frac{1}{24e^{k-\gamma}}\right)\right) \tag{4}$$

for every integer $k > 1$? More generally, for which positive integers N and real numbers α is it true that

$$((e^{k-\alpha})) = \left(\left(e^{k-\alpha} - \frac{1}{Ne^{k-\alpha}}\right)\right) \tag{5}$$

for all integers $k > 1$? If (5) holds for some fixed N and all $k > 1$, must α be transcendental? Let A denote the set of all α for which (5) holds for some fixed integer N and all integers $k > 1$. Is the Lebesgue measure of A equal to zero?

PROOF OF THE THEOREM. We first establish some background information. It is well known that

$$\lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \ln n\right) = \gamma. \tag{6}$$

If we define, for $n \geq 1$,

$$\delta(n) = \sum_{m=1}^{n}\frac{1}{m} - \ln n - \gamma, \tag{7}$$

then it is easily shown that $\delta(n)\downarrow 0$ as $n \to \infty$. An application of the Euler-Maclaurin Summation Formula (see [**1**, p. 444, **4**, pp. 261–262 or **3**, p. 256]) yields the estimate

$$\delta(n) = 1/(2n) - 1/(12n^2) + R_n \tag{8}$$

where $0 < R_n < 1/(120n^4)$.

Now define the rational number r_k by the relation

$$k = 1 + \frac{1}{2} + \cdots + \frac{1}{n_k - 1} + \frac{r_k}{n_k}. \tag{9}$$

Note especially that $0 < r_k < 1$. Using (7) and (9), we obtain the relation

$$\ln n_k = k - \gamma - \delta(n_k) + (1 - r_k)/n_k \tag{10}$$

which may be exponentiated to obtain $n_k = e^{k-\gamma}e^{\beta(n_k)}$, where we have set

$$\beta(n_k) = -\delta(n_k) + \frac{1 - r_k}{n_k} = \left(\frac{1}{2} - r_k\right)\frac{1}{n_k} + \frac{1}{12n_k^2} - R_{n_k}. \tag{11}$$

Consequently, we have

$$n_k - e^{k-\gamma} = e^{k-\gamma}(e^{\beta(n_k)} - 1). \tag{12}$$

We first show that

$$(13) \qquad -1 < n_k - e^{k-\gamma} < +1.$$

Since $\delta(n) > 0$ for all $n \geq 1$, we have

$$\ln(n_k - 1) < \ln(n_k - 1) + \delta(n_k - 1) = S_{n_k - 1} - \gamma < k - \gamma.$$

Exponentiating, we obtain the right-hand side of (13). Since $\ln(n+1) - \ln(n) > 1/(n+1) > \delta(n)$ for all $n \geq 1$, we have

$$\ln(n_k + 1) > \ln(n_k) + \delta(n_k) = S_{n_k} - \gamma > k - \gamma.$$

Exponentiating, we obtain the left-hand side of (13).

Now let $n_k - e^{k-\gamma} = a$. From (12) we obtain the equation

$$(14) \qquad \beta(n_k) = \ln\left(1 + \frac{a}{e^{k-\gamma}}\right).$$

Substituting (11) into (14) and solving for r_k, we get

$$r_k = \frac{1}{2} + \frac{1}{12n_k} - n_k R_{n_k} - a\ln\left(1 + \frac{a}{e^{k-\gamma}}\right) - e^{k-\gamma}\ln\left(1 + \frac{a}{e^{k-\gamma}}\right).$$

Now, since $|a| < 1$, we may write

$$-a\ln\left(1 + \frac{a}{e^{k-\gamma}}\right) = -\frac{a^2}{e^{k-\gamma}} + \frac{a^3}{2e^{2(k-\gamma)}} + \frac{c_1}{e^{3(k-\gamma)}},$$

where $|c_1| \leq 4/9$, and

$$-e^{k-\gamma}\ln\left(1 + \frac{a}{e^{k-\gamma}}\right) = -a + \frac{a^2}{2e^{k-\gamma}} - \frac{a^3}{3e^{2(k-\gamma)}} + \frac{c_2}{e^{3(k-\gamma)}}$$

where $|c_2| \leq 1/3$. Also,

$$(15) \qquad \frac{1}{12n_k} = \frac{1}{12e^{k-\gamma}} - \frac{a}{12e^{2(k-\gamma)}} + \frac{c_3}{e^{3(k-\gamma)}}$$

where $|c_3| \leq 1/9$, and

$$0 < n_k R_{n_k} < \frac{1}{120n_k^3} < \frac{1}{9e^{3(k-\gamma)}}.$$

The bounds on c_1, c_2 and c_3 are independent of k. Hence,

$$(16) \quad r_k = \left(\frac{1}{2} + \frac{1}{12e^{k-\gamma}} + \frac{p}{e^{3(k-\gamma)}}\right) - a\left(1 + \frac{1}{12e^{2(k-\gamma)}}\right) - \frac{a^2}{2e^{k-\gamma}} + \frac{a^3}{6e^{2(k-\gamma)}}$$

where $|p| < 1$. This last expression is a polynomial in a from which the theorem is easily deduced using the inequality $0 < r_k < 1$.

2. A second conjecture. Let us again consider inequality (2) and ask the following question: By how much must the closest partial sums *miss* a given integer k? More precisely, can one determine functions α and β of k alone such that

$$(17) \qquad S_{n_k} - k \geq \alpha(k)$$

and

$$(18) \qquad k - S_{n_k - 1} \geq \beta(k)?$$

Recalling equation (9), we get

(19) $$S_{n_k} - k = (1 - r_k)/n_k$$

and

(20) $$k - S_{n_k - 1} = r_k/n_k.$$

Thus, any lower estimate of the differences in (17) or (18) will involve estimates of r_k and n_k.

Suppose now that the original conjecture about the value of n_k is true; i.e., that $|n_k - e^{k-\gamma}| = |a| < 1/2$. (It was noted earlier in this paper that the conjecture *is* true for values of $k \leq 1000$.) If $a > -1/2$, then equation (16) implies that

(21) $$1 - r_k > \frac{1}{24e^{k-\gamma}} + \frac{1}{16e^{2(k-\gamma)}} + \frac{p}{e^{3(k-\gamma)}},$$

where $|p| < 1$. Combining (15), (17), (19) and (21), we hypothesize that we may choose

$$\alpha(k) = 1/24e^{2(k-\gamma)} + 1/48e^{3(k-\gamma)}$$

and that $1/24$ is the best possible constant here. The author is unable at this time to propose an analogous hypothesis concerning the order of magnitude of $\beta(k)$, but it seems reasonable to suggest that $\beta(k) = O(e^{-2k})$.

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