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ON RANDOM APPROXIMATIONS AND A RANDOM FIXED POINT THEOREM FOR SET VALUED MAPPINGS

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ABSTRACT. We prove a random fixed point theorem in a Banach space for set valued mappings and then derive a corollary that yields a fixed point theorem of Bharucha-Reid and Mukherjea, as a special case.

In a recent paper **[1]**, Bharucha-Reid and Mukherjea proved the following stochastic analogue of the well-known Schauder's fixed point theorem.

THEOREM 1. *Let S be a compact and convex subset of a Banach space E and T: $\Omega \times S \to S$ be a continuous random operator. Then T admits a random fixed point.*

In this paper, we shall show that Theorem 1 can be derived from a more general result. For detailed definitions and terminologies we refer to Bharucha-Reid **[1]** or a recent paper of Itoh **[3]**. Throughout, this paper, (Ω, Σ) is a measurable space with Σ a sigma algebra of subsets of Ω. The symbol 2^E denotes the class of nonempty subsets of a Banach space E. A mapping F: $\Omega \to 2^E$ is called measurable iff for each open set G of E,

$$F^{-1}(G) = \{\omega \in \Omega: F(\omega) \cap G \neq \phi\} \in \Sigma.$$

It may be pointed out that if $F(\omega)$ is compact for each $\omega \in \Omega$, then F is measurable iff $F^{-1}(C) \in \Sigma$ for each closed subset C in E (see Himmelberg **[2]** or Itoh **[3]**).

Let S be a nonempty subset of E. Let T: $\Omega \times S \to 2^E$ be a mapping. T is called

(a) a random operator iff for each fixed $x \in S$, the mapping $T(\cdot, x)$: $\Omega \to 2^E$ is measurable,

(b) upper (lower) semicontinuous (u.s.c, l.s.c) iff for each fixed $\omega \in \Omega$, $T(\omega, \cdot)$: $S \to 2^E$ is u.s.c (l.s.c), that is $(T(\omega, \cdot))^{-1}(C)$ is closed (open) subset of S for each closed (open) subset C of E,

(c) continuous iff T is both u.s.c and l.s.c.

A single valued measurable mapping ϕ: $\Omega \to E$ is a random fixed point of the random operator T: $\Omega \times S \to 2^E$ iff $\phi(\omega) \in T(\omega, \phi(\omega))$ for each $\omega \in \Omega$.

The following selection theorem due to Kuratowski and Ryll-Nardzewski **[4]** is used in the proof of our result.

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PROPOSITION 1. *Let S be a separable closed subset of a Banach space E and let F: $\Omega \to 2^S$ be a measurable function such that $F(\omega)$ is compact for each $\omega \in \Omega$. Then there exists a single valued measurable function ϕ: $\Omega \to S$ with $\phi(\Omega) \in F(\omega)$ for each $\omega \in \Omega$.*

For subsets A and B of a normed space, we shall write

$$d(A, B) = \inf\{\|x - y\| : x \in A, y \in B\}.$$

The following result is a special case of a well-known result of Reich [**5**, Lemma 1.6].

PROPOSITION 2. *Let S be a compact and convex subset of a Banach space E and F: $S \to 2^E$ be a continuous multifunction such that $F(x)$ is compact and convex for each x in S. Then there exists an $x \in S$ with $d(x, Fx) = d(Fx, S)$.*

The main result of this paper is

THEOREM 2. *Let S be a compact and convex subset of E and T: $\Omega \times S \to 2^E$ be a continuous random operator with compact and convex values. Then there exists a single valued measurable map ϕ: $\Omega \to S$ satisfying*

$$d(T(\omega, \phi(\omega)), \phi(\omega)) = d(T(\omega, \phi(\omega)), S),$$

for each $\omega \in \Omega$.

We first prove a few lemmas simplifying the proof of Theorem 2.

LEMMA 1. *Let S be a nonempty subset of a normed space E and T: $S \to 2^E$ be a l.s.c. multifunction. If a sequence $\{x_n\}$ in S converges to an x_0 in S, then for any $y_0 \in Tx_0$, there exists a subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ and a sequence $\{y_{n_i}\}$ with $y_{n_i} \in T(x_{n_i})$ such that $y_{n_i} \to y_0$.*

PROOF. Let for each positive integer n, $N(y_0, 1/n)$ be a neighborhood of y_0 of radius $1/n$. Then $y_0 \in N(y_0, 1/n) \cap Tx_0$. Consequently, for each n, there exists an $\varepsilon_n > 0$ such that $Tz \cap N(y_0, 1/n) \neq \varnothing$ for each $z \in N(x_0, \varepsilon_n) \cap S$. Since $x_n \to x_0$, it follows that for each positive integer k, there exists x_{n_k}, $n_k > n_{k-1}$, such that $Tx_{n_k} \cap N(y_0, 1/k) \neq \varnothing$. If $y_{n_k} \in Tx_{n_k} \cap N(y_0, 1/k)$, then $\{y_{n_k}\}$ satisfies the conclusions of Lemma 1.

LEMMA 2. *Let S be a compact subset of a normed space E and T: $S \to 2^E$ be a continuous multifunction with compact values. Then the real valued functions $f(x) = d(x, Tx)$ and $g(x) = d(Tx, S)$ are continuous on S.*

PROOF. We show that f is continuous. The proof that g is continuous is similar.

Suppose f is not continuous at some point $x_0 \in S$. This implies the existence of an $\varepsilon > 0$ and a subsequence $\{x_n\}$ in S with $x_n \to x_0$ but

$$|f(x_n) - f(x_0)| > \varepsilon \tag{1}$$

for each n. Choose a $y_0 \in Tx_0$ such that $\|y_0 - x_0\| = d(Tx_0, x_0)$. By Lemma 1, there exists a sequence $y_{n_i} \in Tx_{n_i}$ with $y_{n_i} \to y_0$. Thus,

$$f(x_{n_i}) = d(x_{n_i}, Tx_{n_i}) \leqslant \|x_{n_i} - y_{n_i}\| \leqslant \|x_{n_i} - x_0\| + f(x_0) + \|y_0 - y_{n_i}\|.$$

This implies that $\lim(f(x_{n_i}) - f(x_0)) \leqslant \varepsilon$. Without loss of generality, we may assume that $\lim(f(x_n) - f(x_0)) \leqslant \varepsilon$. Now, for each n, choose a $y_n \in Tx_n$ such that $d(x_n, Tx_n) = \|x_n - y_n\|$. Since T is u.s.c, TS is compact and hence $\{y_n\}$ has a subsequence $\{y_{n_i}\} \to y_1 \in Tx_0$ for some y_1. Consequently,

$$f(x_0) = d(x_0, Tx_0) \leqslant \|x_0 - y_1\| \leqslant \|x_0 - x_{n_i}\| + f(x_{n_i}) + \|y_{n_i} - y_1\|.$$

This yields $\lim(f(x_0) - f(x_{n_i})) \leqslant \varepsilon$. Thus $|f(x_{n_i}) - f(x_0)| \leqslant \varepsilon$ eventually. This contradicts (1). Hence f is continuous.

LEMMA 3. *Let S be a nonempty compact subset of a normed vector space E and T: $\Omega \times S \to 2^E$ be a multivalued random operator. Then, for each fixed $x \in S$, the mappings g_x and h_x defined by*

$$g_x(\omega) = d(T(\omega, x), x) \quad \textit{and} \quad h_x(\omega) = d(T(\omega, x), S)$$

are measurable.

PROOF. Let α be a real. Then it is easy to verify that

$$\{\omega \in \Omega: g_x(\omega) < \alpha\} = \{\omega \in \Omega: T(\omega, x) \cap N(x, \alpha) \neq \varnothing\}.$$

This implies that g_x is measurable. To show that h_x is measurable, let D be a countable dense subset of S. Then

$$\{\omega: h_x(\omega) < \alpha\} \equiv \bigcup_{y \in D} \{\omega: d(T(\omega, x), y) < \alpha\}$$
$$= \bigcup_{y \in D} \{\omega: T(\omega, x) \cap N(y, \alpha) \neq \varnothing\}.$$

This implies that h_x is measurable.

PROOF OF THEOREM 2. Define a mapping F: $\Omega \to 2^S$ by

$$F(\omega) = \{x \in S: d(T(\omega, x), x) = d(T(\omega, x), S)\}.$$

Then it follows by Proposition 2 that $F(\omega) \neq \varnothing$. Further by Lemma 2, $F(\omega)$ is closed and hence a compact subset of S for each $\omega \in \Omega$. We show that F is measurable. Let C be a closed subset of S and D a countable dense subset of S. For each n, let $D_n = \{x \in D: d(x, C) < 1/n\}$ and

$$C_n = \bigcup_{x \in D_n} \left\{\omega \in \Omega: d(T(\omega, x), x) < d(T(\omega, x), S) + \frac{1}{n}\right\}.$$

By Lemma 3, C_n is measurable for each n. We show that $F^{-1}(C) = \bigcap_{n=1}^{\infty} C_n$. If $\omega \in F^{-1}(C)$, then $F(\omega) \cap C \neq \varnothing$. This implies that there exists an $x_0 \in C$ with $d(T(\omega, x_0), x_0) = d(T(\omega, x_0), S)$. Since D is dense in S, it follows by Lemma 2, that for each fixed n, there exists an $x_n \in D$ such that $d(x_n, C) < 1/n$ and

$$d(T(\omega, x_n), x_n) \leqslant d(T(\omega, x_0), x_0) + \frac{1}{2n} = d(T(\omega, x_0), S) + \frac{1}{2n}$$
$$\leqslant d(T(\omega, x_n), S) + \frac{1}{n}.$$

Thus $\omega \in \bigcap_{n=1}^{\infty} C_n$. Conversely, if $\omega \in \bigcap_{n=1}^{\infty} C_n$, then for each n, there exists an $x_n \in D_n$ with $d(T(\omega, x_n), x_n) < d(T(\omega, x_n)) + 1/n$. Since $\{x_n\} \subseteq S$ and S is compact, there exists a subsequence $x_{n_i} \to x_0 \in C$. This implies that $d(T(\omega, x_0), x_0) = d(T(\omega, x_0), S)$. Thus $x_0 \in F(\omega) \cap C$, that is, $\omega \in F^{-1}(C)$. This proves that F is measurable. Consequently, by Proposition 1 there exists a single valued measurable function ϕ: $\Omega \to S$ with $\phi(\omega) \in F(\omega)$ for each $\omega \in \Omega$. This yields $d(T(\omega, \phi(\omega)), \phi(\omega)) = d(T(\omega, \phi(\omega)), S)$.

The following, a special case of the above result, contains Theorem 1.

COROLLARY 1. *Under the hypothesis of Theorem* 2, *if in addition* $T(\omega, x) \subseteq S$ *for each* $\omega \in \Omega$, $x \in \partial S$ (*boundary of* S), *then* ϕ *therein in Theorem* 2, *is a random fixed point of* T.

PROOF. If for some $\omega \in \Omega$, $T(\omega, \phi(\omega)) \cap S = \phi$, then $\phi(\omega) \notin \partial S$. Since $\phi(\omega) \in S$, it follows that $\phi(\omega)$ is an interior point of S. Choose a $y \notin T(\omega, \phi(\omega))$ such that $\|y - \phi(\omega)\| = d(T(\omega, \phi(\omega)), \phi(\omega))$. Since $y \notin S$, there exist a λ, $0 < \lambda < 1$ such that $(1 - \lambda)y + \lambda\phi(\omega) \in S$. This implies that

$$\begin{aligned} d(T(\omega, \phi(\omega)), S) &\leqslant \|y - ((1 - \lambda)y + \lambda\phi(\omega))\| \\ &= \lambda\|y - \phi(\omega)\| < d(T(\omega, \phi(\omega)), S). \end{aligned}$$

This is impossible. Thus, for each $\omega \in \Omega$, $T(\omega, \phi(\omega)) \cap S \neq \varnothing$. This implies $d(T(\omega, \phi(\omega)), S) = 0$ for each $\omega \in \Omega$. Hence by Theorem 2, $d(T(\omega, \phi(\omega)), \phi(\omega)) = 0$, that is, $\phi(\omega) \in T(\omega, \phi(\omega))$ for each $\omega \in \Omega$.

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