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HORRORS OF TOPOLOGY WITHOUT AC: A NONNORMAL ORDERABLE SPACE

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ABSTRACT. In the absence of AC there can be a space which is not normal, yet which is orderable and is the topological sum of countably many compact countable metrizable spaces.

1. Introduction. A topological space is called *orderable* if its topology is the order topology of some order. (All orders in this paper are linear orders.) Let LN be the axiom

LN: Every orderable space is normal.

It is well known that AC implies LN [**B**, p. 241, Theorem 11; **HLZ**, 5.4; **L**, §3; **M**; **S**]. Birkhoff has asked if LN can be proved without AC [**B**, p. 252, Problem 83]; this question was repeated in [**HLZ**, 5.7(a) and **Lu**, p. 253]. In this note we show AC is needed for LN. Our result vividly demonstrates the horrors of topology without AC, for we show that in the absence of AC there can be space which is orderable as well as the topological sum of countably many copies of a compact metrizable space K, yet is not normal (hence is not homeomorphic to $\mathbf{N} \times K$).

An obvious question our work suggests is whether LN is properly weaker than AC. We leave this question unanswered, but show LN is equivalent to a weak form of AC.

2. A horrible space. As usual, ζ denotes the order type of $\langle \mathbf{Z}, < \rangle$, where $\mathbf{Z}$ is the set of all integers. Let $\omega\zeta$AC denote the following weak form of AC:

$\omega\zeta$AC: A countable collection $\mathscr{Z}$ has a choice function if there is $\langle <_Z : Z \in \mathscr{Z} \rangle$ such that $\langle Z, <_Z \rangle$ has order type ζ for each $Z \in \mathscr{Z}$.

2.1. THEOREM. *It is consistent with ZF that $\omega\zeta AC$ fails.*

□ As pointed out by Jech, this can be proved the same way one proves that a countable collection of doubletons need not have a choice function [$\mathbf{J_2}$, pp. 198–201, 208]. □

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2.2. THEOREM. *If $\omega\zeta AC$ fails there is a space which is orderable and which is the topological sum of countably many copies of a compact countable metrizable space, yet which is not normal.*

□ We prove the contrapositive. So let $\langle Z_k: k \in \mathbf{Z}\rangle$ and $\langle <_k: k \in \mathbf{Z}\rangle$ be such that $\langle Z_k, <_k\rangle$ has order type ζ for $k \in \mathbf{Z}$. We may assume $Z_k \cap Z_l = \varnothing$ for $k \neq l \in \mathbf{Z}$. One can find $\langle a_k: k \in \mathbf{Z}\rangle$ and $\langle b_k: k \in \mathbf{Z}\rangle$ such that $a_k \neq b_k$ for $k \in \mathbf{Z}$, and if $\overline{Z}_k = \{a_k\} \cup Z_k \cup \{b_k\}$ for $k \in \mathbf{Z}$ then $\overline{Z}_k \cap \overline{Z}_l = \varnothing$ for $k \neq l \in \mathbf{Z}$. Let $L = \bigcup_{k \in \mathbf{Z}} \overline{Z}_k$, and let $<$ order L in the obvious way:

(α) $<$ extends each $<_k$, i.e. $\forall k \in \mathbf{Z}, \forall y, z \in Z_k[y <_k z \Rightarrow y < z]$;

(β) $\forall k \in \mathbf{Z}[a_k < Z_k < b_k$, i.e. $\forall z \in Z_k[a_k < z < b_k]]$;

(γ) $\forall k < l \in \mathbf{Z}[\overline{Z}_k < \overline{Z}_l$, i.e. $\forall y \in \overline{Z}_k \forall z \in \overline{Z}_l[y < z]]$.

Give L the order topology. Clearly a_{k+1} is the immediate $<$-successor of b_k for $k \in \mathbf{Z}$, so

(δ) $\forall k \in \mathbf{Z}[\overline{Z}_k$ is clopen in $L]$.

Hence L is the topological sum of $\{\overline{Z}_k: k \in \mathbf{Z}\}$. Of course, if $\overline{\mathbf{Z}} = \{-\infty\} \cup \mathbf{Z} \cup \{\infty\}$ has the order topology of the obvious order, $\overline{Z}_k$ is homeomorphic to $\overline{\mathbf{Z}}$ for $k \in \mathbf{Z}$. So by our hypothesis, L is normal.

Clearly $A = \{a_k: k \in \mathbf{Z}\}$ and $B = \{b_k: k \in \mathbf{Z}\}$ are disjoint, and because of (δ) they are closed since L, being orderable, is T_1, and since $\forall k \in \mathbf{Z}[|A \cap \overline{Z}_k| = 1 = |B \cap \overline{Z}_k|]$. Let U and V be disjoint neighborhoods of A and B. For each $k \in \mathbf{Z}$ we have $U \cap Z_k \neq \varnothing$, and $U \cap Z_k$ is bounded above in $\langle Z_k, <_k\rangle$ since $V \cap Z_k$ includes a tail of Z_k. Hence the correspondence $k \mapsto \max(U \cap Z_k)$ $(k \in \mathbf{Z})$ is a well-defined choice function for $\langle Z_k: k \in \mathbf{Z}\rangle$. □

2.3. REMARK. If $\langle Z_k: k \in \mathbf{Z}\rangle$, L and $\overline{\mathbf{Z}}$ are as above, the following are equivalent:

(1) $\langle Z_k: k \in \mathbf{Z}\rangle$ has a choice function;

(2) $\langle L, <\rangle$ has order type $(1 + \zeta + 1) \times \zeta$;

(3) L is homeomorphic to $\overline{\mathbf{Z}} \times Z$;

(4) L is metrizable;

(5) L is normal.

3. Equivalents of LN.

3.1. THEOREM. *The following are equivalent*:

(1) *LN*, *i.e. every orderable space is normal*;

(2) *for each complete order the collection of its nonempty convex subsets has a choice function*;

(3) *for each complete order L each pairwise disjoint collection of nonempty convex open subsets has a choice function.*

□ For technical reasons we consider two more statements:

(4) a collection $\mathscr{C}$ of sets has a choice function provided $\mathscr{C}$ admits an order and provided there is $\langle <_C: C \in \mathscr{C}\rangle$ such that $\langle C, <_C\rangle$ is a conditionally complete order for each $C \in \mathscr{C}$;

(5) every completely orderable (self-explanatory) space is hereditarily normal.

The implication (2) ⇒ (3) is obvious. We will prove (5) ⇒ (1) ⇒ (4) ⇒ (2) and (3) ⇒ (5).

PROOF OF (5) $\Rightarrow$ (1). Let the topology of L be induced by $<$. The Dedekind completion $\langle L^+, <^+\rangle$ of $\langle L, <\rangle$ can be constructed in ZF, and the order topology of L coincides with the topology L gets as a subspace of L^+.

PROOF OF (1) $\Rightarrow$ (4). We may assume $\mathscr{C}$ is pairwise disjoint. Let $\lhd$ order $\mathscr{C}$. We may assume each member of $\mathscr{C}$ has an immediate $\lhd$-predecessor and an immediate $\lhd$-successor, for otherwise we replace $\mathscr{C}$ by $\mathscr{C} \times \{0\}$ and enlarge $\mathscr{C} \times \{0\}$ to $\mathscr{C} \times \mathbf{Z}$. But now we can repeat the proof of Theorem 2.2, provided we use "sup" rather than "max" in the definition of the choice function.

(The suggestion to consider the simplest possible nontrivial case, i.e. each $\langle C, <_C\rangle$ has order type ζ, which we used in §2, comes from Arnold Miller.)

PROOF OF (1) $\Rightarrow$ (2). If L is complete, the collection of its nonempty convex sets is

$$\mathscr{C} = \bigcup\{\{(a, b), [a, b), (a, b], [a, b]\} - \{\varnothing\} : a < b \in L\},$$

hence $\mathscr{C}$ can be ordered since $L \times L$ can be ordered. (Of course, $\bigcup\{\{[a, b), (a, b], [a, b]\} : a < b \in L\}$ has a choice function in ZF.)

PROOF OF (3) $\Rightarrow$ (5). We model our proof on Birkhoff's proof of AC $\Rightarrow$ LN [**B**, p. 241, Theorem 11], but give full details because we find that proof vague.

Let L be a space whose topology is induced by a complete order, $<$ say. For technical convenience assume $-\infty, \infty \notin L$, let $\bar{L} = \{-\infty\} \cup L \cup \{\infty\}$ and let $<$ also denote the extension of $<$ to a complete order on $\bar{L}$ in which $-\infty < L < \infty$.

To prove L is hereditarily normal it suffices to prove all open subspaces are normal. So consider any open $X \subseteq L$, and let F, G be disjoint closed subsets of X. Since $-\infty, \infty \notin X$ we can define l(eft) and r(ight): $X \to \bar{L}$ satisfying

(1) $\forall x \in L[r_x < x < l_x]$;

(2) $\forall x < y \in L[(r_x = r_y \text{ and } l_x = l_y) \text{ or } r_x \leqslant l_y \text{ or } r_y \leqslant l_x]$,

by

$$l_x = \max([-\infty, x] - X) \quad \text{and} \quad r_x = \min([x, \infty] - X), \qquad x \in X.$$

Next, let $F^* = G$ and $G^* = F$ and define pairs P by

$$P = \bigcup_{H \in \{F, G\}} \{\langle x, y\rangle \in H \times H^* :$$
$$x < y \text{ and } (x, y) \neq \varnothing \text{ and } (x, y) \cap (F \cup G) = \varnothing\}.$$

Then $\forall \langle x, y\rangle \neq \langle x', y'\rangle \in P[(x, y) \cap (x', y') = \varnothing]$, hence by (3) there is a function m: $P \to L$ which chooses for each $\langle x, y\rangle \in P$ a midpoint $m(x, y) \in (x, y)$.

It should be clear that we now can define $s_x \in [l_x, x)$ for $x \in F \cup G$ (and an auxilliary $\sigma_x \in [l_x, x)$ for some $x \in F \cup G$) by the following:

$$\begin{gathered} \text{Let } H \in \{F, G\} \text{ be such that } x \in H, \text{ then} \\ \text{IF } (l_x, x] \cap H^* = \varnothing \text{ THEN } s_x = l_x, \\ \text{ELSE } \sigma_x = \max((l_x, x] \cap H^*] \text{ and now} \\ \text{IF } (\sigma_x, x) \in P \text{ THEN } s_x = m(\sigma_x, x) \\ \text{ELSE } s_x = \min((\sigma_x, x) \cap H). \end{gathered}$$

For $x \in F \cup G$ also define an analogous $t_x \in (x, r_x)$. Using (2) one easily verifies that the neighborhoods $\bigcup_{x \in F}(s_x, t_x)$ of F and $\bigcup_{x \in G}(s_x, t_x)$ of G are disjoint. $\square$

3.2. COROLLARY. *It is consistent with ZF^- to have $LN + \neg AC$.*

PROOF. If LW is the axiom that every set that admits an order also admits a well-order, then LW implies (2) of 3.1. It is known that it is consistent with ZF^-, i.e. ZF minus foundation, to have LW but not AC [**J₁**, p. 134]. (Unfortunately, LW $\Rightarrow$ AC in ZF [**J₁**, p. 133, Theorem 9.1].) □

Our next corollaries show that for several properties P related to normality, LN holds iff every orderable space has P.

3.3. COROLLARY TO PROOF. *The following can be added to the list of Theorem* 3.1:
(6) *every orderable space is collectionwise normal*;
(7) *every orderable space is collectionwise Hausdorff.*

□ PROOF OF (3) $\Rightarrow$ (6). It suffices to prove that if the topology of L is induced by a complete order and X is open in L then X is collectionwise normal. Well, let $\mathscr{F}$ be a discrete collection of closed sets in X; for $F \in \mathscr{F}$ define $F^* = \bigcup(\mathscr{F} - \{F\})$, and make some obvious little changes in the proof of (3) $\Rightarrow$ (5).

Obviously (6) $\Rightarrow$ (7).

PROOF OF (7) $\Rightarrow$ (4). In the proof of (1) $\Rightarrow$ (4) we only needed to know that in every orderable space every two disjoint closed discrete subsets have disjoint neighborhoods. □

3.4. COROLLARY TO PROOF. *The following can be added to the list of Theorem* 3.1:
(8) *every orderable space satisfies Urysohn's Lemma*;
(9) *every orderable space satisfies the Tietze-Urysohn Extension Theorem.*

□ PROOF. It suffices to prove (9) from (2); along the way we prove (8). Let Y be an orderable space, and let

$$\mathscr{F} = \{\langle F_0, F_1\rangle : F_0 \text{ and } F_1 \text{ are disjoint closed sets in } Y\}.$$

Let L be a completely orderable space in which Y embeds. By the proof of (3) $\Rightarrow$ (5), if there is a choice function for the collection of nonempty convex open subsets of $\mathscr{F}$ then there is a well-defined procedure to find for every $\langle F_0, F_1\rangle \in \mathscr{F}$ two disjoint open subsets U_0, U_1 in $X = L - (\overline{F}_0 \cap \overline{F}_1)$ such that $\overline{F}_i \cap X \subseteq U_i$ for $i \in 2$. Hence there is a function φ from $\mathscr{F}$ to the topology of Y satisfying

$$(*) \qquad \forall\langle F_0, F_1\rangle \in \mathscr{F}\left[F_0 \subseteq \varphi(F_0, F_1) \text{ and } \overline{\varphi(F_0, F_1)} \cap F_1 = \varnothing\right].$$

Of course the existence of such a φ is sufficient to prove without AC that Y satisfies Urysohn's Lemma. In fact it shows there is a function ψ from $\mathscr{F}$ to the set C of continuous functions from Y to $[0, 1]$ such that

$$\forall\langle F_0, F_1\rangle \in \mathscr{F}\left[F_i \subseteq \left(\psi(F_0, F_1)^{\leftarrow}\{i\} \text{ for } i \in 2\right].$$

Of course the existence of such a ψ is sufficient to prove without AC that, for each closed $F \subseteq Y$, each continuous function $f\colon F \to [0, 1]$ extends to a continuous function $\hat{f}\colon Y \to [0, 1]$. □

In this context we remind the reader that (because of the Jech-Sochor embedding theorem [**J₂**, p. 209]) Läuchli [**L**] has shown that it is consistent with ZF that there is a nonsingleton orderable space which is compact (Läuchli's locally compact space can easily be made compact) hence normal, but on which every continuous real valued function is constant.

The proof of Corollary 3.4 suggests the question of whether the function φ on $\mathscr{F}$ can be made to satisfy not only $(*)$ but also

$$\forall \langle F_0, F_1 \rangle, \langle G_0, G_1 \rangle \in \mathscr{F} \left[(F_0 \subseteq G_0 \text{ and } F_1 \supseteq G_1) \Rightarrow \varphi(F_0, F_1) \subseteq \varphi(G_0, G_1) \right],$$

i.e. whether Y is monotonically normal. In other words, we are asking if LN makes all orderable spaces monotonically normal. The proof of [**HLZ**, 5.4 or **Lu**, p. 255] shows LW (as in the proof of Corollary 3.2) is strong enough to make all orderable spaces monotonically normal; however, even though monotone normality is hereditary (so that we only have to worry about compact orderable spaces), (2) of Theorem 3.1 seems too weak.

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