AN n-DIMENSIONAL SUBGROUP OF R^{n+1}
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ABSTRACT. A construction given by R. D. Anderson and J. E. Keisler is modified to show that there exists an n-dimensional subgroup G in R^{n+1} such that dim G^k = n for all k. The group G is connected, locally connected, and divisible.

Introduction. For separable metric spaces the fundamental theorem for the dimension of products is dim X x Y ≤ dim X + dim Y. If X is a continuum with dim X = n, then by [3], dim X x X = 2n or 2n - 1. If X is not a continuum, then it may be that dim X x X = n and in fact dim X^k = n for all k. Anderson and Keisler gave such an example in [1] for each n as a subset of R^{n+1}. In this note we show how to modify the construction of that paper to get X to be a subgroup of R^{n+1} and thus a topological group. The example G_n will have the property that G_n meets every nondegenerate subcontinuum in R^{n+1} and consequently will be connected and locally connected. As a topological group it will have a unique completion which will be R^{n+1} since it is densely embedded in R^{n+1} [2, Theorem 1, p. 248]. It is also true that the unique topological group completion of G_n will be R^k(n+1).

The group G_n cannot contain a continuum since a nondegenerate continuum A has the property that dim A^k ≥ k. There is no complete separable metric space X which is connected and locally connected such that dim X^k = dim X for all k unless dim X = 0 or dim X = ∞. The reason for this is that X would either be degenerate or contain an arc.

Notation. Let card(A) be the cardinality of the set A. Let c denote the cardinality of the reals which we think of as an initial ordinal. We denote an s-dimensional hyperplane in R^r by H^s. For i = 1, 2, let H_i be a hyperplane of dimension t_i in R^r. Then H_1 and H_2 are in general position with respect to each other if, whenever H_1' and H_2' are translations of H_1 and H_2 with H_1' ∩ H_2' ≠ ∅, then H_1' ∩ H_2' = H' where t = max{0, t_1 + t_2 - r}.

Let Q denote the rational numbers and a = (r_1, ..., r_s) ∈ Q^s. Let H_a be the hyperplane in R^{n+s} defined by H_a = {(r_1x, ..., r_sx)|x ∈ R^n}. If a ≠ 0 in Q^s, then H_a is an n-dimensional hyperplane. There are countably many such hyperplanes since Q^s is countable.

The construction. The purpose of this paper is to give a construction proving the following main theorem.
MAIN THEOREM. For each positive integer \( n \) there is a subgroup \( G_n \) in \( \mathbb{R}^{n+1} \) such that \( \dim G_n = n \) and \( \dim G_k^n = n \) for all positive integers \( k \). The group \( G_n \) is also connected, locally connected, and divisible.

The proof is patterned after [1]. However, it is as easy to give a complete proof here as to assume familiarity with that proof. We repeat three lemmas from [1] without proof.

**LEMMA 1.** Let \( K \) be a subset of \( \mathbb{R}^n \) such that \( K \cap C \neq \emptyset \) for every nondegenerate continuum \( C \) in \( \mathbb{R}^n \). Then \( \dim K \leq n - 1 \).

**LEMMA 2.** Given a countable collection of hyperplanes \( \{H_i\}_{i=1}^{\infty} \), a \( k \)-sphere \( S \), and a hyperplane \( H \), all in \( \mathbb{R}^r \), such that \( S - H = U_1 \cup U_2 \) where \( p \in U_1 \) with \( U_1 \) open and closed in \( S - H \), and \( U_1 \cap U_2 = \emptyset \), then there exists a hyperplane \( H' \) such that (1) \( \dim H' = k \), (2) for each positive integer \( i \), \( H' \) is in general position with respect to \( H_i \), and (3) \( S - H' = V_1 \cup V_2 \) where \( p \in V_1 \subset U_1 \) is open and closed in \( S - H \) and \( V_1 \cap V_2 = \emptyset \).

In \( \mathbb{R}^{ns} \) choose a countable dense set of points and \((ns - 1)\)-dimensional spheres \( S^{ns-1} \) with rational radius about them such that none of them contains the origin. For each \( S^{ns-1} \) choose a countable set of \((ns - 1)\)-dimensional hyperplanes \( H^{ns-1} \) such that their complementary domains form a basis for the topology of \( S^{ns-1} \) and such that each \( H^{ns-1} \) is in general position with respect to each \( H_a \) for all \( a \in Q^s \). This is possible by Lemma 2. For each of the countably many \( S^{ns-1} \)'s, choose countably many \( S^{ns-2} \)'s by \( S^{ns-1} \cap H^{ns-1} \) for the \( H^{ns-1} \) chosen above.

Inductively, for each \( S^{ns-k} = S^{ns-k+1} \cap H^{ns-k+1} \), choose a countable set of hyperplanes \( H^{ns-k} \) whose complementary domains in \( S^{ns-k} \) form a basis for the topology of \( S^{ns-k} \) such that each \( H^{ns-k} \) is in general position with respect to each \( H_a \) for all \( a \in Q^s \). Let \( S_i = S_i^{ns-n} \) be the countably many \((ns - n)\)-spheres that are obtained when \( k = n \).

**LEMMA 3.** Let \( T \subset \mathbb{R}^{ns} \) be such that, for each \( i \), \( T \cap S_i = \emptyset \). Then \( \dim T \leq n - 1 \).

This construction is the same as in [1], except that the hyperplanes \( H^{ns-k} \) are in general position with respect to a different family of hyperplanes \( \{H_a|a \in Q^s\} \) rather than the family \( \gamma \) in [1].

**Proof of the Main Theorem.** We first prove a special case of the Main Theorem. We show that for a fixed \( s \) there is a subgroup \( G_n \subset \mathbb{R}^{n+1} \) such that \( \dim G_n = n = \dim G_n^s \). We will then indicate how to modify the proof so that \( \dim G_n^s = n \) for all positive integers.

**Case 1.** For a fixed positive integer \( s \), \( \dim G_n^s = n \).

Let \( \{C_\alpha|\alpha < c\} \) be an enumeration of the nondegenerate subcontinua in \( R^{n+1} \) and assume \( 0 \in C_0 \). We want \( G_n \) to be such that \( G_n \cap C_\alpha \neq \emptyset \) for all \( \alpha \) and \( G_n^s \cap Y = \emptyset \), where \( Y = \bigcup_{i=1}^{\infty} S_i \) and the \( S_i \)'s are the \(((n + 1)s - (n + 1))\)-spheres in \( R^{(n+1)s} \) described just before Lemma 3.

Let \( G_0 = \{0\} \). Then suppose that \( G_\beta \) has been chosen for all \( \beta < \alpha < c \) with the properties that (1) \( G_\beta \) is a divisible subgroup of \( R^{n+1} \); (2) \( G_\beta \subset G_\gamma \) for all \( \beta < \gamma < \alpha \); (3) \( G_\beta^s \cap Y = \emptyset \) for all \( \beta < \alpha \); (4) \( G_\beta \cap C_\alpha \neq \emptyset \); and (5) \( \text{card } G_\beta \leq 8_0 \cdot \text{card } ([0, \alpha]) \) for all \( \beta < \alpha \). Then let \( G'_\alpha = \bigcup_{\beta < \alpha} G_\beta \). Then \( G'_\alpha \) will
satisfy (1)–(3) and (5). If \( G'_\alpha \cap C_\alpha \neq \emptyset \), then let \( G_\alpha = G'_\alpha \) and all five properties are satisfied for \( \{G_\beta\}_{\beta < \alpha - 1} \). If \( G'_\alpha \cap C_\alpha = \emptyset \), then we extend the group \( G'_\alpha \) in a manner which we now describe. Let

\[
A = \bigcup \{ \pi_k(Q \cdot (H_\alpha \cap (S_i + (G'_\alpha)^s))) | a \in Q^s, i \in N, \text{ and } k \in \{1, \ldots, s\} \}.
\]

Note that for each fixed \( a \in Q^s, i \in N, \) and \( k \in \{1, \ldots, s\} \), \( \pi_k(Q \cdot (H_\alpha \cap (S_i + (G'_\alpha)^s))) \) has cardinality at most \( N_0 \cdot \text{card}(G'_\alpha) \) since \( H_\alpha \cap (S_i + (g_1, \ldots, g_s)) \) is at most two points for all \( (g_1, \ldots, g_s) \in (G'_\alpha)^s \). This implies that \( \text{card} A < c \). This implies that one can choose \( \rho_\alpha \in C_\alpha - A \). Then we let \( G_\alpha = G'_\alpha + Q \cdot \rho_\alpha \). Note that \( \text{card} G_\alpha \leq N_0 \cdot \text{card}(\{0, \alpha + 1\}) \), as required. Suppose that \( G^s_\alpha \cap Y \neq \emptyset \). Then there is an \( a = (r_1, \ldots, r_s) \in Q^s \) and \( (g_1, \ldots, g_s) \in (G'_\alpha)^s \) such that \( (r_1, \ldots, r_s) \rho_\alpha + (g_1, \ldots, g_s) \in G^s_\alpha \cap Y \). Clearly, some \( r_i \neq 0 \) or \( (g_1, \ldots, g_s) \in Y \) and \( G^s_\beta \cap Y \neq \emptyset \) for some \( \beta < \alpha \), a contradiction. Now this implies that we have \( \rho_\alpha \in S_k + (G'_\alpha)^s \) for some \( k \) and thus \( \rho_\alpha = \pi_i(\rho_\alpha / r_i) \in \pi_i(Q \cdot (H_\alpha \cap (S_k + (G'_\alpha)^s))) \). This implies that \( \rho_\alpha \in A \), a contradiction. Therefore, \( G^s_\alpha \cap Y = \emptyset \) and \( \{G^s_\beta\}_{\beta < \alpha + 1} \) satisfies (1)–(5).

Let \( G_n = \bigcup_{\alpha < c} G_\alpha \). Then \( G_n \) will be a divisible subgroup of \( R^{n+1} \), \( G_n \cap C_\alpha \neq \emptyset \) for all \( \alpha < c \), and \( G^s_\alpha \cap Y = \emptyset \). Thus \( \dim G_n \geq n \) and \( \dim G^s_n \leq n \). Thus, \( \dim G_n = n = \dim G^s_n \). This proves the special case for a fixed \( s \).

**Case 2.** Construct \( G_n \) such that \( \dim G^s_n = n \) for all \( s \).

The construction is similar to Case 1. For each positive integer \( s \), let \( Y_s = \bigcup_{i=1}^{\infty} S_i \) where each \( S_i \) is an \( [(n + 1)s - (n + 1)] \)-sphere in \( R^{(n+1)s} \) as in Case 1. Then we can construct \( \{G_\alpha\}_{\alpha < c} \) as in Case 1 with (1') \( G_\alpha \) a divisible subgroup of \( R^{n+1} \); (2') \( G_\alpha \subseteq G_\beta \) for all \( \alpha < \beta < c \); (3') \( G^s_\alpha \cap Y_s = \emptyset \) for all \( \alpha < c \) and all positive integers \( s \); (4') \( G_\alpha \cap C_\alpha \neq \emptyset \) for all \( \alpha < c \); and (5') \( \text{card} G_\alpha < c \) for all \( \alpha < c \). Then \( G_n = \bigcup_{\alpha < c} G_\alpha \) will be the required divisible subgroup of \( R^{n+1} \). The strengthening of (3) to (3') is straightforward and we leave this to the reader.

**COROLLARY.** There is a divisible subgroup \( G_n \) in \( R^{n+1} \) such that \( \dim G_n = \dim G^s_n = n \).

**PROOF.** This follows from Lemma 4 of [1], since \( \dim G^s_n = n \) for all \( s \).

**REFERENCES**


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