AN n-DIMENSIONAL SUBGROUP OF $R^{n+1}$

JAMES KEESLING

ABSTRACT. A construction given by R. D. Anderson and J. E. Keisler is modified to show that there exists an $n$-dimensional subgroup $G$ in $R^{n+1}$ such that $\dim G^k = n$ for all $k$. The group $G$ is connected, locally connected, and divisible.

Introduction. For separable metric spaces the fundamental theorem for the dimension of products is $\dim X \times Y \leq \dim X + \dim Y$. If $X$ is a continuum with $\dim X = n$, then by [3], $\dim X \times X = 2n$ or $2n - 1$. If $X$ is not a continuum, then it may be that $\dim X \times X = n$ and in fact $\dim X^k = n$ for all $k$. Anderson and Keisler gave such an example in [1] for each $n$ as a subset of $R^{n+1}$. In this note we show how to modify the construction of that paper to get $X$ to be a subgroup of $R^{n+1}$ and thus a topological group. The example $G_n$ will have the property that $G_n$ meets every nondegenerate subcontinuum in $R^{n+1}$ and consequently will be connected and locally connected. As a topological group it will have a unique completion which will be $R^{n+1}$ since it is densely embedded in $R^{n+1}$ [2, Theorem 1, p. 248]. It is also true that the unique topological group completion of $G_n^k$ will be $R^k(n+1)$.

The group $G_n$ cannot contain a continuum since a nondegenerate continuum $A$ has the property that $\dim A^k \geq k$. There is no complete separable metric space $X$ which is connected and locally connected such that $\dim X^k = \dim X$ for all $k$ unless $\dim X = 0$ or $\dim X = \infty$. The reason for this is that $X$ would either be degenerate or contain an arc.

Notation. Let $\text{card}(A)$ be the cardinality of the set $A$. Let $\mathfrak{c}$ denote the cardinality of the reals which we think of as an initial ordinal. We denote an $s$-dimensional hyperplane in $R^r$ by $H^s$. For $i = 1, 2$, let $H_i$ be a hyperplane of dimension $t_i$ in $R^r$. Then $H_1$ and $H_2$ are in general position with respect to each other if, whenever $H_1'$ and $H_2'$ are translations of $H_1$ and $H_2$ with $H_1' \cap H_2' \neq \emptyset$, then $H_1' \cap H_2' = H^t$ where $t = \max\{0, t_1 + t_2 - r\}$.

Let $Q$ denote the rational numbers and $a = (r_1, \ldots, r_s) \in Q^s$. Let $H_a$ be the hyperplane in $R^{ns}$ defined by $H_a = \{(r_1x, \ldots, r_sx) | x \in R^n\}$. If $a \neq 0$ in $Q^s$, then $H_a$ is an $n$-dimensional hyperplane. There are countably many such hyperplanes since $Q^s$ is countable.

The construction. The purpose of this paper is to give a construction proving the following main theorem.
Main Theorem. For each positive integer \( n \) there is a subgroup \( G_n \) in \( \mathbb{R}^{n+1} \) such that \( \dim G_n = n \) and \( \dim G^k_n = n \) for all positive integers \( k \). The group \( G_n \) is also connected, locally connected, and divisible.

The proof is patterned after [1]. However, it is as easy to give a complete proof here as to assume familiarity with that proof. We repeat three lemmas from [1] without proof.

Lemma 1. Let \( K \) be a subset of \( \mathbb{R}^n \) such that \( K \cap C \neq \emptyset \) for every nondegenerate continuum \( C \) in \( \mathbb{R}^n \). Then \( \dim K \geq n - 1 \).

Lemma 2. Given a countable collection of hyperplanes \( \{H_i\}_{i=1}^\infty \), a \( k \)-sphere \( S \), and a hyperplane \( H \), all in \( \mathbb{R}^r \), such that \( S - H = U_1 \cup U_2 \) where \( p \in U_1 \) with \( U_1 \) open and closed in \( S - H \), and \( U_1 \cap U_2 = \emptyset \), then there exists a hyperplane \( H' \) such that (1) \( \dim H' = k \), (2) for each positive integer \( i \), \( H' \) is in general position with respect to \( H_i \), and (3) \( S - H' = V_1 \cup V_2 \) where \( p \in V_1 \subset U_1 \) is open and closed in \( S - H \) and \( V_1 \cap V_2 = \emptyset \).

In \( \mathbb{R}^{ns} \) choose a countable dense set of points and \( (ns - 1) \)-dimensional spheres \( S^{ns-1} \) with rational radius about them such that none of them contains the origin. For each \( S^{ns-1} \) choose a countable set of \( (ns - 1) \)-dimensional hyperplanes \( H^{ns-1} \) such that their complementary domains form a basis for the topology of \( S^{ns-1} \) and such that each \( H^{ns-1} \) is in general position with respect to each \( H_a \) for all \( a \in Q^s \). This is possible by Lemma 2. For each of the countably many \( S^{ns-1} \)'s, choose countably many \( H^{ns-2} \)'s by \( S^{ns-1} \cap H^{ns-2} \) for the \( H^{ns-1} \) chosen above.

Inductively, for each \( S^{ns-k} = S^{ns-k+1} \cap H^{ns-k+1} \), choose a countable set of hyperplanes \( H^{ns-k} \) whose complementary domains in \( S^{ns-k} \) form a basis for the topology of \( S^{ns-k} \) such that each \( H^{ns-k} \) is in general position with respect to each \( H_a \) for all \( a \in Q^s \). Let \( S_i = S_i^{ns-n} \) be the countably many \( (ns - n) \)-spheres that are obtained when \( k = n \).

Lemma 3. Let \( T \subset \mathbb{R}^{ns} \) be such that, for each \( i \), \( T \cap S_i = \emptyset \). Then \( \dim T \leq n - 1 \).

This construction is the same as in [1], except that the hyperplanes \( H^{ns-k} \) are in general position with respect to a different family of hyperplanes \( \{H_a|a \in Q^s\} \) rather than the family \( \gamma \) in [1].

Proof of the Main Theorem. We first prove a special case of the Main Theorem. We show that for a fixed \( s \) there is a subgroup \( G_n \subset \mathbb{R}^{n+1} \) such that \( \dim G_n = n = \dim G^*_n \). We will then indicate how to modify the proof so that \( \dim G^*_n = n \) for all positive integers.

Case 1. For a fixed positive integer \( s \), \( \dim G^*_n = n \).

Let \( \{C_\alpha|\alpha < c\} \) be an enumeration of the nondegenerate subcontinua in \( \mathbb{R}^{n+1} \) and assume \( 0 \in C_0. \) We want \( G_n \) to be such that \( G_n \cap C_\alpha \neq \emptyset \) for all \( \alpha \) and \( G^*_n \cap Y = \emptyset \), where \( Y = \bigcup_{i=1}^\infty S_i \) and the \( S_i \)'s are the \( ((n+1)s-(n+1)) \)-spheres in \( \mathbb{R}^{(n+1)s} \) described just before Lemma 3.

Let \( G_0 = \{0\} \). Then suppose that \( G_\beta \) has been chosen for all \( \beta < \alpha < c \) with the properties that (1) \( G_\beta \) is a divisible subgroup of \( \mathbb{R}^{n+1} \); (2) \( G_\beta \subset G_\gamma \) for all \( \beta < \gamma < \alpha \); (3) \( G^*_\beta \cap Y = \emptyset \) for all \( \beta < \alpha \); (4) \( G_\beta \cap C_\beta \neq \emptyset \); and (5) \( \text{card } G_\beta \leq n_0 \cdot \text{card } ([0, \alpha]) \) for all \( \beta < \alpha \). Then let \( G^*_\alpha = \bigcup_{\beta < \alpha} G_\beta \). Then \( G^*_\alpha \) will
satisfy (1)–(3) and (5). If \( G'_\alpha \cap C_\alpha \neq \emptyset \), then let \( G_\alpha = G'_\alpha \) and all five properties are satisfied for \( \{G_\beta\}_{\beta < \alpha + 1} \). If \( G'_\alpha \cap C_\alpha = \emptyset \), then we extend the group \( G'_\alpha \) in a manner which we now describe. Let

\[
A = \bigcup \{ \pi_k(Q \cdot (H_\alpha \cap (S_i + (G'_\alpha)^s))) \mid a \in Q^s, i \in N, \text{ and } k \in \{1, \ldots, s\} \}.
\]

Note that for each fixed \( a \in Q^s, i \in N, \text{ and } k \in \{1, \ldots, s\} \), \( \pi_k(Q \cdot (H_\alpha \cap (S_i + (G'_\alpha)^s))) \) has cardinality at most \( n_0 \cdot \text{card}(A) \) since \( H_\alpha \cap (S_i + (G'_\alpha)^s) \) is at most two points for all \( (g_1, \ldots, g_s) \in (G'_\alpha)^s \). This implies that \( \text{card}(A) < c \). Then we let \( G_\alpha = G'_\alpha + Q \cdot p_\alpha \). Note that \( G_\alpha \leq n_0 \cdot \text{card}(A) \), as required. Suppose that \( G_\alpha \cap Y \neq \emptyset \). Then there is an \( a = (r_1, \ldots, r_s) \in Q^s \) and \( (g_1, \ldots, g_s) \in (G'_\alpha)^s \) such that \( p_\alpha = \pi_k((a + (g_1, \ldots, g_s)) \in (G'_\alpha)^s) \). This implies that \( G_\alpha \cap Y \neq \emptyset \). Clearly, some \( r_1 \neq 0 \) or \( (g_1, \ldots, g_s) \in Y \) and \( G_\alpha \cap Y \neq \emptyset \). This implies that we have \( a = (r_1, \ldots, r_s) \in Q^s \) and \( (g_1, \ldots, g_s) \in (G'_\alpha)^s \). Then \( G_\alpha \cap Y = \emptyset \) and \( G_\alpha \cap Y \neq \emptyset \). Thus \( \text{dim} G_n \geq n \) and \( \text{dim} G_n^s \leq n \). Thus, \( \text{dim} G_n = n = \text{dim} G_n^s \). This proves the special case for a fixed \( s \).

Case 2. Construct \( G_n \) such that \( \text{dim} G_n^s = n \) for all \( s \).

The construction is similar to Case 1. For each positive integer \( s \), let \( Y_s = \bigcup_{i=1}^{\infty} S_i \) where each \( S_i \) is an \( [(n + 1)s - (n + 1)] \)-sphere in \( R^{(n+1)s} \) as in Case 1. Then we can construct \( \{G_\alpha\}_{\alpha < c} \) as in Case 1 with \((1')\) \( G_\alpha \) a divisible subgroup of \( R^{n+1} \); \((2')\) \( G_\alpha \subset G_\beta \) for all \( \alpha < \beta < c \); \((3')\) \( G_\alpha \cap Y_s = \emptyset \) for all \( \alpha < c \) and all positive integers \( s \); \((4')\) \( G_\alpha \cap C_\alpha \neq \emptyset \) for all \( \alpha < c \); and \((5')\) \( \text{card} G_\alpha < c \) for all \( \alpha < c \). Then \( G_n = \bigcup_{\alpha < c} G_\alpha \) will be the required divisible subgroup of \( R^{n+1} \). The strengthening of (3) to (3') is straightforward and we leave this to the reader.

COROLLARY. There is a divisible subgroup \( G_n \) in \( R^{n+1} \) such that \( \text{dim} G_n = \text{dim} G_n^s = n \).

PROOF. This follows from Lemma 4 of [1], since \( \text{dim} G_n^s = n \) for all \( s \).

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611