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HAUSDORFF HYPERCUBES WHICH DO NOT CONTAIN ARCLESS CONTINUA

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ABSTRACT. A Hausdorff arc is a compact connected Hausdorff space with exactly two noncut points. The finite product of a Hausdorff arc is called a Hausdorff hypercube. Suppose that X is a Hausdorff arc which is first countable at none of its points and n is a positive integer. We show that every nondegenerate subcontinuum of X^n contains a Hausdorff arc. Thus X^n contains no nondegenerate hereditarily indecomposable continuum.

During our study of nonmetric hereditarily indecomposable continua we have discovered that if X is a Hausdorff arc which is not first countable at any of its points, then the Hausdorff hypercube, X^n for n a positive integer, does not contain a nondegenerate hereditarily indecomposable continuum[1]. We later discovered that the argument generalizes to a proof that, if X is a Hausdorff arc which is not first countable at any of its points, then every nondegenerate subcontinuum of X^n for n a positive integer contains a Hausdorff arc. A Hausdorff arc is defined as a compact connected Hausdorff space with exactly two noncut points. However, Grispolakis and Tymchatyn [**GT**] have shown that each Hausdorff continuum of $\dim \geq 2$ contains an indecomposable continuum.

Let us introduce some notation. Suppose that X is a Hausdorff arc with endpoints x_0 and x_1. Then there is an order relation $<$ on X in the order from x_0 to x_1 and the order topology induces the topology of X. Suppose $a \in X$ and $b \in X$. Then let

$$[a, b] = \{x \mid a \leq x \leq b\}, \qquad (a, b) = \{x \mid a < x < b\}.$$

Suppose $x \in X$. Then we say that x is a type I point if some countable infinite sequence of points converges to x and x is a type II point if no countable infinite sequence of points converges to x.

THEOREM 1. *There exists a Hausdorff arc X which is first countable at none of its points.*

PROOF. Consider ω_1, the first uncountable ordinal. Let X be the collection of all functions from ω_1 into $[0,1]$. Define an order relation on X as follows: If $f \in X$ and $g \in X$ let $f <_X g$ if and only if $f(\alpha) < g(\alpha)$ with respect to the order on $[0,1]$, where α is the first ordinal $\lambda < \omega_1$ such that $f(\lambda) \neq g(\lambda)$. The order $<_X$ is a generalized lexicographic ordering. It is not difficult to prove that X with the order topology induced by $<_X$ is compact, connected, and linearly ordered; hence X is a Hausdorff arc. Suppose now that X is first countable at some point $x = \{x(\alpha)\}_{\alpha<\omega_1}$. Let us assume that x is not an endpoint of X, only a slight

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modification of the following argument is needed in case x is an endpoint. Then there is a countable sequence $\{(a_i, b_i)\}_{i=1}^{\infty}$ which forms a basis at x, $a_i < x < b_i$. Let α_i be the first ordinal λ so that $a_i(\lambda) \neq x(\lambda)$ and let β_i be the first ordinal λ so that $b_i(\lambda) \neq x(\lambda)$. So we have $a_i(\alpha_i) < x(\alpha_i)$ and $x(\beta_i) < b_i(\beta_i)$. Thus since $\{\lambda | \lambda = \alpha_i$ or $\lambda = \beta_i$ for some positive integer $i\}$ is countable there is an ordinal δ so that $\alpha_i < \delta$ and $\beta_i < \delta$ for all positive integers i. Either $x(\lambda) \neq 0$ for uncountably many λ or $x(\lambda) \neq 1$ for uncountably many λ. Let us suppose without loss of generality that $x(\lambda) \neq 0$ for uncountably many λ. Let δ' be such that $\delta < \delta'$ and $x(\delta') \neq 0$. Let $y \in X$ be defined as follows:

$$\begin{aligned} y(\lambda) &= x(\lambda) \quad \text{if } \lambda < \delta', \\ y(\lambda) &= 0 \quad \text{if } \delta' \leq \lambda. \end{aligned}$$

Then $y \neq x$ and, since $y(\lambda) = x(\lambda)$ for $\lambda < \delta'$, $y \in (a_i, b_i)$ for all i. Thus $\{(a_i, b_i)\}_{i=1}^{\infty}$ is not a basis at x, and we have a contradiction. Thus if $x \in X$ then X is not first countable at x.

LEMMA 2.1. *Suppose X is a Hausdorff arc and $A_1, A_2, \ldots$ is an infinite sequence of points in X^n. Then some subsequence has a sequential limit point.*

PROOF. If $a_1, a_2, \ldots$ is an infinite subsequence of X then there is either an infinite increasing subsequence of $\{a_i\}_{i=1}^{\infty}$ or an infinite decreasing subsequence of $\{a_i\}_{i=1}^{\infty}$. So, in either case, there is an infinite subsequence of $\{a_i\}_{i=1}^{\infty}$ with sequential limit point. Let $A_i = (a_i^1, a_i^2, \ldots, a_i^n)$. Let us also assume without loss of generality that $\{a_i^j\}_{i=1}^{\infty}$ is infinite for $j = 1$. Then some infinite subsequence $\{a_{n_i}^1\}_{i=1}^{\infty}$ has a sequential limit point. So some infinite subsequence of $\{a_{n_i}^2\}_{i=1}^{\infty}$ has a sequential limit. And by induction we can get a sequence of integers $\{l_i\}_{i=1}^{\infty}$ so that $\{a_{l_i}^1\}_{i=1}^{\infty}$ is infinite and has a sequential limit and $\{a_{l_i}^j\}_{i=1}^{\infty}$ has a sequential limit $j = 1, 2, 3, \ldots, n$. Therefore, $\{A_{l_i}\}_{i=1}^{\infty}$ is a subseqence of $\{A_i\}_{i=1}^{\infty}$ with a sequential limit.

LEMMA 2.2. *Suppose X is a Hausdorff arc. Then if $M_1, M_2, \ldots$ is a countable sequence of closed subsets of X and M is the limiting set of the sequence, and $M_i \cap M = \emptyset$ for all i, then each point of M is of type* I.

PROOF. Suppose $x \in M$ and x is a type II point. If i is a positive integer then there exists an open set (r_i, s_i) so that $r_i < x < s_i$ and $(r_i, s_i) \cap M_i = \emptyset$. Since x is a type II point and $r_i < x$ for all i then there exists a point $r \in X$ so that $r_i < r < x$ for all i. Similarly, there is a point $s \in X$ so that $x < s < s_i$ for all i. Thus, (r, s) is an open set containing x but no point of $\bigcup_{i=1}^{\infty} M_i$. This is a contradiction.

LEMMA 2.3. *If X is a Hausdorff arc which is first countable at none of its points then the set of type* I *points of x is dense in X and the set of type* II *points is dense in X.*

PROOF. It is sufficient to prove the theorem to show that every such arc contains a type I point and a type II point. Suppose then that $X = [a, b]$ is a Hausdorff arc which is first countable at none of its points and X is ordered from a to b with respect to $<$. Let $\{x_i\}_{i=1}^{\infty}$ be a countably infinite sequence of points of X. Then by Lemma 2.1, some subsequence has a sequential limit point which would be a type I point.

Now let us establish that X contains a type II point. Let G be the set to which g belongs if and only if $g = [r, s]$ for some $r < s$ in X. Let G be well ordered. By transfinite induction a subcollection J of G may be constructed so that

(1) the first element of G belongs to J,

(2) if $j \in J$ and $L(j) = \{g \in J \mid g < j\}$ then j is the first element of G which is a proper subset of every element of $L(j)$ and which has no endpoint in common with any element of $L(j)$, and

(3) J is maximal with respect to properties (1) and (2) above.

Then, since J is maximal, $\bigcap\{j \mid j \in J\}$ is a degenerate set $\{z\}$. We claim that z is a type II point. Since X is not first countable at z and z is the endpoint of no element of J then no countable set is cofinal in J. Suppose that z is not a type II point. Then let $x_1, x_2, \ldots$ be a sequence of points distinct from z which has sequential limit z. For each integer n there exist an element $j_n \in J$ so that $x_n \notin j_n$. Let $I = \bigcap_{n=1}^{\infty} j_n$, then $x_n \notin I$ for all positive integers n. But z is a limit point of $\{x_i\}_{i=1}^{\infty}$ so z must be an endpoint of I. But X is not first countable so no countable set is cofinal in J and there is a first element j' in J which follows every element of the set $\{j_n\}_{n=1}^{\infty}$. By the definitions of J and I, $j' \subset I$. Let j'' be the first element of J that follows j'. Then $z \in j''$ and j'' intersects neither endpoint of j', hence neither endpoint of I. This is a contradiction. Therefore z is a type II point.

Let $\pi_i \colon X^n \to X$ be the natural projection map, so if $x = (x^1, x^2, \ldots, x^n)$ then $\pi_i(x) = x^i$.

LEMMA 2.4. *Suppose X is an arc, n is a positive integer, and M is a nondegenerate subcontinuum of X^n so that if K is a subcontinuum of M, k is an integer, $1 \le k \le n$, and $r < s$ in X, then the number of components of $K \cap (X^{k-1} \times [r, s] \times X^{n-k-1})$ that intersect both $K \cap (X^{k-1} \times \{r\} \times X^{n-k-1})$ and $K \cap (X^{k-1} \times \{s\} \times X^{n-k-1})$ is finite. Then M contains an arc.*

PROOF. We prove the lemma by induction on n the number of factors of X^n. Thus let m be an integer so that if $n < m$ and M is a subcontinuum of X^n which satisfies the hypothesis then M contains an arc. Suppose M is a subcontinuum of X^n which satisfies the hypothesis of the lemma but which contains no arc. For some k, $1 \le k \le m$, $\pi_k(M)$ is nondegenerate. Without loss of generality we assume that $k = 1$. Let r and s be such that $[r, s] \subset \pi_1(M)$.

Suppose K is a subcontinuum of M and $u < v$. Then let $K_{[u,v]} = K \cap ([u, v] \times X^{m-1})$ and $K_u = K \cap (\{u\} \times X^{m-1})$. Therefore, by the induction hypothesis, if K is a subcontinuum of M then K_u must be totally disconnected. (Otherwise, K_u contains a copy of a nondegenerate subcontinuum of X^{m-1} which satisfies the hypothesis of the lemma.)

Let I be a subcontinuum of M irreducible from M_r to M_s. Let $p \in I - M_r \cup M_s$. Let Γ be an index set such that if $r < a < \pi_1(p) < b < s$ then $a = a_\gamma$ and $b = b_\gamma$ for some $\gamma \in \Gamma$. Let $r < a_\gamma < p < b_\gamma < s$ and let L_γ be the component of $I_{[a_\gamma, b_\gamma]}$ that contains p. If $H \subset I$ let $\mathrm{Int}_I(H)$ denote the interior of H with respect to I and let $\mathrm{Bd}_I(H)$ denote the boundary of H with respect to I.

Claim a. Suppose $q \in L_\gamma$ for some γ, $a_\gamma < \pi_1(q) < b_\gamma$. Then $q \in \mathrm{Int}_I(L_\gamma)$.

PROOF. Suppose not. Then there are infinitely many components of $I_{[a_\gamma, b_\gamma]}$. Let $\{L_i\}_{i \in A}$, where A is some infinite index set, be an infinite sequence of such components so that q is a limit point of $\bigcup_{i\in A} L_i$. But then there exists a number t such that either $\pi_1(q) < t$ and $I_{[t,b_\gamma]}$ has infinitely many components intersecting both I_t and I_{b_t} or $t < \pi_1(q)$ and $I_{[a_\gamma,t]}$ has infinitely many components intersecting both I_{a_γ} and I_t. This contradicts the fact that M satisfies the hypothesis of the lemma. This proves Claim a.

For $\gamma \in L$ let $J_\gamma \subset I$ be a continuum irreducible from I_r to L_γ and let $K_\gamma \subset I$ be a continuum irreducible from I_s to L_γ.

Claim b. $J_\gamma \cap L_\gamma \subset I_{a_\gamma} \cup I_{b_\gamma}$ and $K_\gamma \cap L_\gamma \subset I_{a_\gamma} \cup I_{b_\gamma}$.

PROOF. Suppose $J_\gamma \cap L_\gamma \not\subset I_{a_\gamma} \cup I_{b_\gamma}$ and $q \in J_\gamma \cap L_\gamma - I_{a_\gamma} \cup I_{b_\gamma}$. So $a_\gamma < \pi_1 q < b_\gamma$ and, by Claim a, $q \in \mathrm{Int}_I L_\gamma$. But since J_γ is irreducible from I_r to L_γ, every point of $J_\gamma \cap L_\gamma$ is a limit point of $J_\gamma - J_\gamma \cap L_\gamma$, which is a contradiction. Similarly, $K_\gamma \cap L_\gamma \subset I_{a_\gamma} \cup I_{b_\gamma}$.

Therefore, we have $I = J_\gamma \cup L_\gamma \cup K_\gamma$, and since $p \notin J_\gamma \cup K_\gamma$ we also have $J_\gamma \cap K_\gamma = \emptyset$.

Claim c. J_γ is unique and K_γ is unique.

PROOF. Suppose that J_γ is not unique and that J is a subcontinuum of I irreducible from I_r to L_γ which is distinct from J_γ. Suppose that J contains a point z not in J_γ. Then J contains a point z' not in $L_\gamma \cup J_\gamma$. The continuum J must also satisfy Claim b, so $J \cap K_\gamma = \emptyset$. So $z' \notin L_\gamma \cup J_\gamma \cup K_\gamma$. But then $J_\gamma \cup L_\gamma \cup K_\gamma$ is a proper subcontinuum of I which intersects I_r and I_s, which is a contradiction. Similarly, J_γ cannot contain a point not in J. A similar argument proves that K_γ is unique. This proves Claim c.

Claim d. If γ_1 and γ_2 are such that $a_{\gamma_1} < a_{\gamma_2} < \pi_1(p) < b_{\gamma_2} < b_{\gamma_1}$, then $J_{\gamma_1} \subset \mathrm{Int}_I(J_{\gamma_2})$ and $K_{\gamma_1} \subset \mathrm{Int}_I(K_{\gamma_2})$.

PROOF. Since J_{γ_1} and J_{γ_2} are unique we have $J_{\gamma_1} \subset J_{\gamma_2}$. Since $I = J_{\gamma_i} \cup L_{\gamma_i} \cup K_{\gamma_i}$ then, by Claim b, $\mathrm{Bd}_I(J_{\gamma_i}) = J_{\gamma_i} \cap L_{\gamma_i} \subset (I_{a_{\gamma_i}} \cup I_{b_{\gamma_i}})$. Also by definition of L_γ we have $L_{\gamma_2} \subset L_{\gamma_1}$. So $L_{\gamma_2} \cap L_{\gamma_1} = L_{\gamma_2}$. So

$$\begin{aligned}(\mathrm{Bd}_I J_{\gamma_2}) \cap J_{\gamma_1} &= (J_{\gamma_2} \cap L_{\gamma_2}) \cap J_{\gamma_1} \\ &= (J_{\gamma_2} \cap L_{\gamma_2} \cap L_{\gamma_1} \cap J_{\gamma_1}) \\ &\subset (I_{a_{\gamma_2}} \cup I_{b_{\gamma_2}}) \cap (I_{a_{\gamma_1}} \cup I_{b_{\gamma_1}}) = \emptyset.\end{aligned}$$

Therefore, $J_{\gamma_1} \subset \mathrm{Int}_I J_{\gamma_2}$. Similarly, $K_{\gamma_1} \subset \mathrm{Int}_I K_{\gamma_2}$. This proves Claim d.

Claim e. $I - \{p\} = (\bigcup_{\gamma\in\Gamma} J_\gamma) \cup (\bigcup_{\gamma\in\Gamma} K_\gamma)$ and $\bigcup_{\gamma\in\Gamma} J_\gamma$ and $\bigcup_{\gamma\in\Gamma} K_\gamma$ are mutually separated.

PROOF. By Claim a, $p \in \mathrm{Int}_I(L_\gamma)$ for all $\gamma \in A$ so $p \notin J_\gamma$ and $p \notin K_\gamma$ for all $\gamma \in A$. Suppose $\bigcup_{\gamma\in\Gamma} J_\gamma$ and $\bigcup_{\gamma\in\Gamma} K_\gamma$ are not mutually separated and q is a point of $\bigcup_{\gamma\in\Gamma} J_\gamma$ which is a limit point of $\bigcup_{\gamma\in\Gamma} K_\gamma$. Then $q \in J_\alpha$ for some $\alpha \in \Gamma$. There exists $\beta \in \Gamma$ such that $a_\alpha < a_\beta < \pi_1(p)$. So $J_\alpha \subset \mathrm{Int}_I J_\beta$ and $J_\beta \cap \bigcup_{\gamma\in\Gamma} K_\gamma = \emptyset$. So q is not a limit point of $\bigcup_{\gamma\in\Gamma} K_\gamma$. Similarly, $\bigcup_{\gamma\in\Gamma} K_\gamma$ does not contain any limit points of $\bigcup_{\gamma\in\Gamma} J_\gamma$.

Suppose $x \in I - \{p\} - I_r \cup I_s$ but $x \notin (\bigcup_{\gamma\in\Gamma} J_\gamma) \cup (\bigcup_{\gamma\in\Gamma} K_\gamma)$. By the induction hypothesis, $I_{\pi_1(p)}$ is totally disconnected. So whether or not $\pi_1(x) = \pi_1(p)$, there exist $\gamma \in \Gamma$ and a_γ and b_γ so that $x \notin L_\gamma$. So then $x \in J_\gamma \cup K_\gamma$. Therefore, $I - \{p\} = (\bigcup_{\gamma\in\Gamma} J_\gamma) \cup (\bigcup_{\gamma\in\Gamma} K_\gamma)$. This proves Claim e.

Therefore every point p of I with $\pi(p) \notin \{r, s\}$ is a cut point of I. So I must contain a Hausdorff arc. This establishes Lemma 2.4.

THEOREM 2. *If X is a Hausdorff arc and X is not first countable at any of its points and n is a positive integer, then every nondegenerate subcontinuum of X^n contains an arc.*

PROOF. We shall now prove the theorem by induction. Clearly the theorem is true if $n = 1$. Suppose then that the theorem is true for all integers less than n. Suppose that $\hat{M}$ is a nondegenerate subcontinuum of X^n which contains no arc. By Lemma 2.4 there exists a subcontinuum M of $\hat{M}$, an integer k, and elements r and s in X so that the set of components of $M \cap (X^{k-1} \times [r, s] \times X^{n-k})$ that intersect both $M \cap (X^{k-1} \times \{r\} \times X^{n-k})$ and $M \cap (X^{k-1} \times \{s\} \times X^{n-k})$ is infinite. Without loss of generality we can assume that $k = 1$. Also, if $r \leq \hat{r} < \hat{s} \leq s$ then the set of components of $M \cap ([\hat{r}, \hat{s}] \times X^{n-1})$ that intersect both $\{\hat{r}\} \times X^{n-1}$ and $\{\hat{s}\} \times X^{n-1}$ is infinite. Thus, let us assume that $[r, s]$ has been chosen so that r is a type II point. Let $C_1, C_2, \ldots$ be an infinite subsequence of components of $([r, s] \times X^{n-1}) \cap M$ that intersect both $(\{r\} \times X^{n-1}) \cap M$ and $(\{s\} \times X^{n-1}) \cap M$. Let $x_i \in C_i \cap (\{r\} \times X^{n-1})$. Then some subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ has a sequential limit point x. So $\{C_{n_i}\}_{i=1}^{\infty}$ has a limiting set C and C must be a continuum. Also $C \cap (\{s\} \times X^{n-1}) \cap M \neq \emptyset$ and $C \cap (\{r\} \times X^{n-1}) \cap M \neq \emptyset$, and C is a subcontinuum of M. Also, we may assume that $C_{n_i} \cap C = \emptyset$ for all i since $C \cap C_j \neq \emptyset$ for at most one integer j. Let $x_i = (x_i^1, x_i^2, \ldots, x_i^n)$ so that $x_i^1 = r$ for all positive integers i. Further, there is an integer $2 \leq l \leq n$ so that the set $\{x_i^l\}_{i=1}^{\infty}$ is infinite. Let us suppose that $\{x_i\}_{i=1}^{\infty}$ has been chosen so that $x_i^l = x_j^l$ if and only if $i = j$. Let $\{O_i\}_{i=1}^{\infty}$ be a sequence of basic open sets so that $O_i = \prod_{m=1}^{n}(u_i^m, v_i^m)$, $x_i \in O_i$, $\overline{O}_i \cap C = \emptyset$, and $[u_i^l, v_i^l] \cap [u_j^l, v_j^l] = \emptyset$ for all $i \neq j$. Thus $\overline{O}_i \cap \overline{O}_j = \emptyset$ for all $i \neq j$. Let a_i^l, b_i^l be such that $u_i^l < a_i^l < x_i^l < b_i^l < v_i^l$. Let R_i be the basic open set

$$R_i = \prod_{m=1}^{l-1} (u_i^m, v_i^m) \times (a_i^l, b_i^l) \times \prod_{m=l+1}^{n} (u_i^m, v_i^m)$$

and $x_i \in R_i$. Let L_i be the component of $C_i \cap \overline{R}_i$ containing x_i, so $L_i \subset ([r, s] \times X^{n-1}) \cap M$ and $L_i \cap \mathrm{Bd}(\overline{R}_i) \neq \emptyset$. By the induction hypothesis, every subcontinuum of $M \cap (\{r\} \times X^{n-1})$ is degenerate. So $L_i \cap (\{r\} \times X^{n-1})$ must be totally disconnected, and $\pi_1(L_i)$ is nondegenerate because if $\pi_1(L_1)$ were degenerate then $\pi_1(L_1)$ would be homeomorphic to a nondegenerate subcontinuum of X^{n-1} and hence would contain an arc. Let L be the limiting set of $\{L_i\}_{i=1}^{\infty}$, thus $L \subset C$. Let K be the limiting set of $\{\pi_l(L_i)\}_{i=1}^{\infty}$, then $\pi_l(L) \subset K$. For each i, $\pi_l(L_i) \subset [a_i^l, b_i^l] \subset (u_i^l, v_i^l)$ and $\pi_l(L_j) \cap (u_i^l, v_i^l) = \emptyset$ whenever $i \neq j$, so no point of $\pi_l(L_i)$ lies in K. So by Lemma 2.2, every point of K is a type I point; but $\pi_l(L) \subset K$, so every point of $\pi_l(L)$ is a type I point; but $\pi_l(L)$ is connected, so by Lemma 2.3, $\pi_l(L)$ is degenerate.

For each i the set $\pi_i(L_i)$ is nondegenerate and $r \in \pi_1(L_i)$, so there exists $q_i > r$ so that $[r, q] \subset \pi_1(L_i)$. But r is a type II point so no subsequence of $\{q_i\}_{i=1}^{\infty}$ converges to r and hence there is a point q such that $r < q < q_i$ for all i. So $[r, q] \subset \pi_1(L_i)$ for all i, so $\pi_1(L)$ is nondegenerate. Thus

$$L \subset \prod_{i=1}^{l-1} X \times \{\pi_l(L)\} \times \prod_{i=l+1}^{n} X.$$

Thus L is a nondegenerate continuum which embeds in X^{n-1}. But by the induction hypothesis L contains an arc, this is a contradiction. So the theorem is established.

Since no hereditarily indecomposable continuum contains an arc, we have

COROLLARY. *If X is a Hausdorff arc and X is not first countable at any of its points then, for each positive integer n, X^n does not contain a hereditarily indecomposable continuum.*

REMARK. Thus Bing's theorem [**B**] that every two-dimensional metric continuum contains a hereditarily indecomposable continuum does not generalize to Hausdorff continua.

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