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A NOTE ON INTERSECTION OF LOWER SEMICONTINUOUS MULTIFUNCTIONS

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ABSTRACT. Let F_1 and F_2 be closed and convex valued multifunctions from a topological space X to a normed space Y. Assume that the multifunctions are lower semicontinuous at x_0. We proof that the intersection multifunction $F = F_1 \cap F_2$ is lower semicontinuous at x_0 provided $F(x_0)$ is bounded and has nonempty interior.

1. Introduction. Let F be a multifunction from a topological space X to a uniform space $(Y, \mathscr{U})$, i.e. F is a mapping from X to the family of all subsets of Y. F will be called lower semicontinuous (lsc) at $x_0 \in X$ if for every $V \in \mathscr{U}$ there is $U \in N(x_0)$ such that $x \in U$ implies $F(x_0) \subset V(F(x))$, where $N(x_0)$ stands for the neighbourhood filter of x_0 and, for $A \subset Y$, $V(A) = \{y \in Y\colon (a, y) \in V \text{ for some } a \in A\}$. Such multifunctions will also be called Hausdorff-lower semicontinuous (H-lsc). Accordingly, if $(Y, \|\cdot\|)$ is a normed space then F is lsc at x_0 if and only if for every $\varepsilon > 0$ there is $U \in N(x_0)$ such that $F(x_0) \subset F(x) + B_\varepsilon$ for every $x \in U$, where $B_\varepsilon = \{y \in Y\colon \|y\| < \varepsilon\}$. Note that F is lsc at x_0 if and only if the multifunction $\overline{F}$, i.e. $F(x) = \overline{F(x)}$ for all $x \in x$, is lsc at x_0.

Let us recall the usual concept of lower semicontinuity. A multifunction F from a topological space X to a topological space Y is said to be Vietoris-lower semicontinuous (V-lsc) at $x_0 \in X$ if, for every open $G \subset Y$ with $F(x_0) \cap G \neq \varnothing$, there is $U \in N(x_0)$ such that $x \in U$ implies $F(x) \cap G \neq \varnothing$. It is known **[6]** that F is V-lsc at x_0 if and only if it is continuous at x_0 as a mapping from X to the hyperspace of all subsets of Y equipped with the lower Vietoris topology. If Y is a uniform space and F is H-lsc at x_0 then it is V-lsc at x_0. The converse also holds if the set $F(x_0)$ is totally bounded **[6]**.

It is well known that neither H-lsc nor V-lsc are preserved under finite intersections of multifunctions. And, unlike upper semicontinuity **[3, 5]** no compactness type assumptions are helpful in this context. The classical result of Kuratowski **[5**, p. 180**]** says that the multifunction $F = F_1 \cap F_2$ is V-lsc at x_0 provided F_1 is V-lsc at x_0 and F_2 is constant, being equal, for every $x \in X$, to a fixed open subset of Y. Other results on the intersection of V-lsc multifunctions can be found in **[8, 7, 3** and **1]**.

In this note we provide sufficient conditions for Hausdorff-lower semicontinuity of intersection of multifunctions. Our result improves an earlier result of one of the

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authors obtained in [**10**] for finite-dimensional spaces. The key of the proof is an application of the well-known cancellation law for sets in topological vector spaces ([**9**], see also [**11**]): Let A, B and C be subsets of a real topological vector space. If B is bounded, and C is nonempty closed and convex, then $A + B \subset \overline{C + B}$ implies $A \subset C$.

2. Auxiliary lemmas. In the remaining part of this paper $Y = (Y, \|\cdot\|)$ is assumed to be a real normed space.

LEMMA 1. *If A is a convex bounded subset of Y and* $\operatorname{int} A \neq \varnothing$, *then for every* $\varepsilon > 0$ *there are a set* $C \subset \operatorname{int} A$ *and* $\delta > 0$ *such that* $C + B_\delta \subset A \subset C + B_\varepsilon$.

PROOF. Take an arbitrary $\varepsilon > 0$. Without loss of generality we can assume that $0 \in \operatorname{int} A$. Since A is bounded, $\lambda \operatorname{int} A \subset B_{\varepsilon/2}$ for some $0 < \lambda < 1$. Moreover, there is $\delta > 0$ such that $B_\delta \subset \lambda \operatorname{int} A$. Thus, putting $C = (1 - \lambda)\operatorname{int} A$ we get $C + B_\delta \subset A \subset C + B_\varepsilon$, because $\overline{A} = \overline{\operatorname{int} A}$.

The following example shows that the assumption of the boundedness of A cannot be omitted in the above lemma.

EXAMPLE. Let $Y = l^\infty$ and put $A = \{(t_k) \in l^\infty\colon t_1 \geqslant 0 \text{ and } t_k \leqslant k(1 - t_1) \text{ for } k \geqslant 2\}$. Then A is convex and $\operatorname{int} A \neq \varnothing$. Take $\varepsilon = \frac{1}{2}$ and suppose that there are $C \subset \operatorname{int} A$ and $\delta > 0$ such that

$$(+) \qquad C + B_\delta \subset A \subset C + B_{1/2}.$$

For $n \in \mathbf{N}$ let us put $t_k^n = 0$ if $k \neq n$, $k \in \mathbf{N}$ and $t_n^n = n$. Then $x_n = (t_k^n)_{k \in \mathbf{N}} \in A$ for all $n \in \mathbf{N}$, so by $(+)$ for every $n \in \mathbf{N}$ there is $y_n = (s_k^n)_{k \in \mathbf{N}} \in C$ such that $\|x_n - y_n\| = \sup\{|t_k^n - s_k^n|\colon k \in \mathbf{N}\} < \frac{1}{2}$. Take $0 < \alpha < \min\{\delta, \frac{1}{2}\}$. Then $y_n + z \in C + B_\delta \subset A$ for every $n \in \mathbf{N}$, where $z = (\alpha, \alpha, \ldots)$. It follows that $s_1^n + \alpha > 0$ and $\alpha + s_n^n \leqslant n(1 - s_1^n - \alpha)$ for $n \geqslant 2$, hence $\alpha n \leqslant \frac{1}{2} - \alpha$ for $n \geqslant 2$, a contradiction.

However, if Y is a finite-dimensional space then Lemma 1 can be strengthened.

LEMMA 2. *Let A be a convex subset of $\mathbf{R}^n$ with nonempty interior. Then for every* $\varepsilon > 0$ *there are a set* $C \subset \operatorname{int} A$ *and* $\delta > 0$ *such that* $C + B_\delta \subset A \subset C + B_\varepsilon$.

PROOF. Assume that A is unbounded. Otherwise, we can apply Lemma 1. It is clear that the lemma holds if $n = 1$. Suppose then that the thesis of the lemma is satisfied for every convex subset $D \subset \mathbf{R}^{n-1}$ with nonempty interior. Take an arbitrary $\varepsilon > 0$ and consider two cases:

1°. *A contains a line.* Without loss of generality we can assume that A contains the x_n-axis. Putting D to be the image of $\operatorname{int} A$ by the projection into $\mathbf{R}^{n-1}$ we have $\operatorname{int} A = D \times \mathbf{R}$. Thus there are $E \subset \operatorname{int} D$ and $\delta > 0$ such that $E + (B_\delta \cap \mathbf{R}^{n-1}) \subset D \subset E + (B_\varepsilon \cap \mathbf{R}^{n-1})$. Then denoting by C the set $C = E \times \mathbf{R}$ we get $C \subset \operatorname{int} A$ and $C + B_\delta \subset A \subset C + B_\varepsilon$.

2°. *A does not contain a line.* We can suppose that $\operatorname{int} A$ contains the nonnegative part of the x_n-axis and that for some $\lambda_0 > 0$ the set $A_1 = A \cap \{(x, \mu)\colon x \in \mathbf{R}^{n-1}$ and $\mu \leqslant \lambda_0\}$ is bounded and has nonempty interior. By Lemma 1 there are $G_1 \subset \operatorname{int} A_1$ and $\alpha > 0$ such that $C_1 + B_\alpha \subset A_1 \subset C_1 + B_\varepsilon$. Let M denote the hyperplane $M = \{(x, \lambda_0): x \in \mathbf{R}^{n-1}\}$. Since $D = A \cap M$ is a convex body in an $(n-1)$-dimensional space, there are $E \subset \operatorname{int} D$ and $\beta > 0$ such that $E + (B_\beta \cap M) \subset D \subset E + (B_{\varepsilon/\sqrt{2}} \cap M)$. Put $A_2 = A \cap \{(x, \mu): x \in \mathbf{R}^{n-1}$ and $\mu \geqslant \lambda_0\}$. Then taking $0 < \sigma < \min\{\beta, \varepsilon\sqrt{2}/4\}$ we get the following: For every $y \in \partial A_2$ there exists $z \in \operatorname{int} A_2$, such that $\|z - y\| \leqslant \varepsilon/2$ and $z + B_\sigma \subset \operatorname{int} A_2$, where ∂A_2 denotes the boundary of A_2. Let C_2 denote the set $C_2 = \{y \in A_2: \inf\{\|z - y\|: z \in A_2\} \geqslant \sigma\}$. Let us observe that $C_2 + B_\sigma \subset A_2 \subset C_2 + B_\varepsilon$. Consequently, putting $C = C_1 \cup C_2$ and taking $0 < \delta < \min\{\alpha, \sigma\}$ we get $C + B_\delta \subset A \subset C + B_\varepsilon$.

A multifunction F from X to Y is called locally convex-valued (locally closed-valued) at $x_0 \in X$ if there is $U \in N(x_0)$ such that $F(x)$ is convex (closed) for all $x \in U$. The following lemma is proved in [**10**].

LEMMA 3. *Assume that a multifunction F from X to Y is lsc and locally convex-valued at $x_0 \in X$. If* $\operatorname{int} F(x_0) \neq \varnothing$ *then* $\operatorname{int} \cap \{\overline{F(x)}: x \in U\} \neq \varnothing$ *for some $U \in N(x_0)$.*

3. Main results.

THEOREM A. *Assume that the multifunctions F_1 and F_2 from X to Y are locally closed-valued and locally convex-valued at $x_0 \in X$. If F_1 and F_2 are lsc at x_0 and the set $F(x_0) + F_1(x_0) \cap F_2(x_0)$ is bounded and* $\operatorname{int} F(x_0) \neq \varnothing$ *then the multifunction $F = F_1 \cap F_2$ is lsc at x_0.*

PROOF. Let $\varepsilon > 0$ be arbitrary. By Lemma 1 there are a subset $C \subset \operatorname{int} F(x_0)$ and $\delta > 0$ such that $C + B_\delta \subset F(x_0) \subset C + B_\varepsilon$. Since F_1 and F_2 are lsc at x_0, there is $U \in N(x_0)$ such that $F_i(x_0) \subset F_i(x) + B_\delta$ for all $x \in U$ and $i = 1, 2$. Without loss of generality we can assume that F_1 and F_2 are closed and convex-valued on U. Thus, applying the cancellation law we get $C \subset F(x) = F_1(x) \cap F_2(x)$ for every $x \in U$. But it follows that $F(x_0) \subset C + B_\varepsilon \subset F(x) + B_\varepsilon$ for all $x \in U$.

THEOREM B. *Let $Y = \mathbf{R}^n$ and assume that the multifunctions F_1 and F_2 are locally convex-valued at $x_0 \in X$. If F_1 and F_2 are lsc at x_0 and* $\operatorname{int} F(x_0) \neq \varnothing$ *then the multifunction $F = F_1 \cap F_2$ is lsc at x_0.*

PROOF. Applying Lemma 2 and proceeding as in the proof of Theorem A we obtain that the multifunction $\overline{F_1} \cap \overline{F_2}$ is lsc at x_0. Then, by Lemma 3 we find $U \in N(x_0)$ such that F_1 and F_2 are convex-valued on U and $\operatorname{int}(\overline{F_1(x)} \cap \overline{F_2(x)}) \neq \varnothing$ for all $x \in U$. Then, since Y is finite dimensional, we have $\operatorname{int}(F_1(x) \cap F_2(x)) \neq \varnothing$ and therefore $\overline{F_1(x)} \cap \overline{F_2(x)} = \overline{F_1(x) \cap F_2(x)}$, whenever $x \in U$ (see e.g. [**2**, p. 253]). Hence, the multifunction $\overline{F_1 \cap F_2}$, and so also F, is lsc at x_0.

4. Counterexamples. We give some examples concerning Theorems A and B. The first one shows that the assumption $\operatorname{int} F(x_0) \neq \varnothing$ cannot be omitted.

EXAMPLE 1. Let $Y = \mathbf{R}^2$, $F_1(x) = \operatorname{conv}\{(0,0), (1,0), (0,-1)\}$ and $F_2(x) = \operatorname{conv}\{(0,x), (0,1), (1,0)\}$ for all $x \in [0,1]$. Then F_1 and F_2 are compact and convex-valued, lsc at every $x \in [0,1]$ but $F = F_1 \cap F_2$ is not lsc at 0. Note that F is nonempty valued and $\operatorname{int} F(0) = \varnothing$.

The second example shows that both multifunctions F_1 and F_2 must be locally convex-valued.

EXAMPLE 2. Let $Y = \mathbf{R}^2$, $F_1(x) = \operatorname{conv}\{(0, x), (0, 1), (1, 0), (\frac{1}{2}, 0)\} \cup \operatorname{conv}\{(\frac{1}{2}, 0), (1, 0), (1, -1)\}$ and $F_2(x) = \operatorname{conv}\{(0, 0), (1, 0), (1, -1)\}$ for all $x \in [0, 1]$. Then F_1 and F_2 are compact-valued and lsc at every $x \in [0, 1]$. F_2 is convex-valued while F_1 is not. F is not lsc at 0.

The third example shows that the boundedness of $F(x_0)$ in Theorem A is essential.

EXAMPLE 3. Let $Y = l^\infty$ and $F_1(x) = \{(t_k) \in l^\infty\colon t_1 \geqslant x$ and $t_k \leqslant k - x$ for $k \geqslant 2\}$ and $F_2(x) = \{(t_k) \in l^\infty\colon t_1 \leqslant 1 - x$ and $t_k \leqslant k(1 - t_1 - x)$ and $t_k \leqslant k + t_1/k - x/k$ for $k \geqslant 2\}$ for all $x \in [0, 1]$. Then F_1 and F_2 are closed and convex-valued. Moreover, they are lsc at 0. The set $F(0) = \{(t_k) \in l^\infty\colon 0 \leqslant t_1 \leqslant 1$ and $t_k \leqslant k(1 - t_1)$ for $k \geqslant 2\}$ has nonempty interior but is not bounded. F is not lsc at 0.

Finally, the last example shows that in all infinite-dimensional normed spaces the multifunctions in Theorem A must be locally closed-valued.

EXAMPLE 4. Let Y be an infinite-dimensional normed space and let f be a linear noncontinuous functional on Y. Put $A = \{y \in B_1\colon f(y) < 0\} \cup \{0\}$ and $B = \{y \in B_1\colon f(y) > 0\} \cup \{0\}$ where B_1 is the closed unit ball of Y. Then $\overline{A} = \overline{B} = B_1$. Let us define the multifunctions F_1 and F_2 as follows: $F_1(0) = F_2(0) = B_1$ and $F_1(x) = A$ and $F_2(x) = B$ for all $x \in (0, 1]$. Then F_1 and F_2 are lsc and convex-valued. The multifunction F is nonempty valued, the set $F(0)$ is bounded with nonempty interior but F is not lsc at 0.

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