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A FURTHER RESULT ON EXTENDING EXPANSIVE HOMEOMORPHISMS

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ABSTRACT. In this note we give a necessary and sufficient condition for a self-homeomorphism defined on an open subset of a compact metric space to be extended to the whole space using the concept of a generator. Two examples of homeomorphic extensions of expansive homeomorphisms are given, one examining the equivalence of the theorem and the other showing that a previous result's sufficient condition for the extension of expansive homeomorphisms is not necessary.

1. Introduction. Since 1962 there have been works by Bryant [**2**], Williams [**7**], and Wine [**10**] wherein the extension of expansive homeomorphism was investigated. The result of this note gives a necessary and sufficient condition for the extension of an expansive homeomorphism using the concept of a generator introduced by Keynes and Robertson [**3**].

DEFINITION 1. A homeomorphism h of a metric space (X, ρ) onto itself is *expansive* with expansive constant δ if given any two distinct points x and y of X there is an integer n such that $\rho(h^n(x), h^n(y)) > \delta$.

DEFINITION 2. Let h be a homeomorphism of the compact metric space (X, ρ) onto itself. A finite open cover $\mathscr{U}$ of X is a *generator* for (X, h) if for each bisequence $\{A_i\}$ of elements of $\mathscr{U}$, the intersection $\bigcap_{i=-\infty}^{\infty} h^{-i}(\overline{A_i})$ is at most one point.

DEFINITION 3. If h is a homeomorphism of the metric space (X, ρ) onto itself and x is a point of X, then the *orbit* of x is $\bigcup\{h^n(x) \mid n \text{ is an integer}\}$. Notation is $O(x)$.

DEFINITION 4. For a homeomorphism of the metric space (X, ρ) onto itself, the set $\{x_\alpha \in X \mid \alpha \in A\}$ is an *orbital basis* of (X, ρ) with respect to h if $\bigcup\{O(x_\alpha) \mid \alpha \in A\} = X$ and α not equal to β implies $O(x_\alpha)$ not equal to $O(x_\beta)$.

The following theorem, characterizing expansiveness is found in [**3** and **4**].

THEOREM 1. *Let h be a homeomorphism of the compact metric space (X, ρ) onto itself. Then h is expansive if and only if there is a generator for (X, h).*

2. Result. We now use Theorem 1 to prove the following result concerning the extension of expansive homeomorphisms.

THEOREM 2. *Let (X, d) be an open subspace of the compact metric space (Y, ρ), let h be an expansive homeomorphism of (X, d) onto itself with expansive constant δ, and let*

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$\hat{h}$ be a homeomorphic extension of h to Y such that $\hat{h}$ restricted to $Z = Y - X$ is expansive with expansive constant δ. Then $\hat{h}$ will be expansive on Y with expansive constant δ if and only if every generator of $(Z, \hat{h})$ which has the form $\mathcal{U} = \{A_\alpha \cap Z | A_\alpha$ is open in Y, $\operatorname{diam}(A_\alpha) < \delta$, $\alpha \in \mathcal{A}\}$ has the property that, for every bisequence of elements of $\mathcal{U}$, $\{A_i \cap Z\}_{i=-\infty}^{\infty}$, $\bigcap_{i=-\infty}^{\infty} \hat{h}^{-i}(\bar{A}_i)$ is at most one point.

PROOF. *Necessity.* Suppose $\hat{h}$ is expansive on Y, and that $\mathcal{U}$ has the form given in the statement of the theorem. Extend $\{A_\alpha | A_\alpha$ is used in $\mathcal{U}\}$ to an open cover $\mathcal{G}$ of Y, where

$$\mathcal{G} = \left\{ G_\beta | \operatorname{diam}(G_\beta) < \delta, \beta \in \mathcal{B} \right\}$$

and G_β not in $\mathcal{U}$ implies G_β is disjoint from Z. Let $\mathcal{G}^*$ be a finite subcover of $\mathcal{G}$. Return to $\mathcal{G}^*$ any of the A_α in $\mathcal{U}$ that may have been removed from $\mathcal{G}$ by the taking of the finite subcover, and call the new cover $\mathcal{H}$. The cover $\mathcal{H}$ is finite with all elements having diameter less than δ. Since $\hat{h}$ is expansive on Y with expansive constant δ, the cover $\mathcal{H}$ is a generator for $(Y, \hat{h})$. Therefore, $\mathcal{H}$ has the property that for any bisequence $\{A_i\}_{i=-\infty}^{\infty}$ of elements of $\mathcal{H}$ the intersection $\bigcap_{i=-\infty}^{\infty} \hat{h}^{-i}(\bar{A}_i)$ consists of at most one point. Since $\mathcal{U}$ is a subset of $\mathcal{H}$, it must be the case that $\mathcal{U}$ has the desired property also.

Sufficiency. Let $\mathcal{G}^*$ be an open cover of Y by the open balls of radius $\delta/2$ and let $\mathcal{G}$ be a finite subcover of $\mathcal{G}^*$. Let $\mathcal{U}$ be the subset of $\mathcal{G}$ which consists of all elements of $\mathcal{G}$ which have nonvoid intersection with $Z = Y - X$. Then the family $\hat{\mathcal{U}} = \{G_\lambda \cap Z | G_\lambda$ is in $\mathcal{U}\}$ is a generator for $\hat{h}$ restricted to Z, since $\hat{h}$ is expansive on Z with expansive constant δ.

Now let $S = \{A_k\}_{k=-\infty}^{\infty}$ be any bisequence from $\mathcal{G}$ and suppose that the intersection $\bigcap\{\hat{h}^{-k}(\bar{A}_k) | A_k$ is in $S\}$ contains distinct points x and y. By hypothesis, if S is a subset of $\mathcal{U}$, the intersection would contain at most one point. Therefore, it must be the case that at least one of the A_k in S must not be in $\mathcal{U}$. Hence, the intersection must be contained in X. However, h is expansive on X with expansive constant δ, so there exists an integer n such that $d(h^n(x), h^n(y)) > \delta$. This implies that $\hat{h}^n(x)$ and $\hat{h}^n(y)$ cannot both be elements of the set $\bar{A}_n$. A contradiction has been reached, and it must be that the intersection can contain at most one point. Hence $\mathcal{G}$ is a generator for $\hat{h}$ on Y, and $\hat{h}$ is expansive on Y with expansive constant δ.

3. Examples. In **[10]** the following theorem is proved. We give an example which shows that the condition given in this theorem is not necessary.

THEOREM 3. *Let (X, ρ) be a subspace of the metric space (Y, d) and let h be an expansive homeomorphism of (X, ρ) onto itself with expansive constant δ. Suppose $\hat{h}$ is a homeomorphic extension of h to (Y, d). Then f is expansive with expansive constant δ if*

(1) *f restricted to $Y - X$ is expansive with expansive constant δ, and*

(2) *there is an orbital basis B of X with respect to h such that $d(x, Y - X) > \delta$ for every x in B.*

EXAMPLE 1. Let $S = \{u | u = 1 - 1/k, k = 2, 3, 4, \dots\} \cup \{u | u = 1/k, k = 2, 3, 4, \dots\}$, let $X = S \times S$, and let (X, ρ) be X considered as a subspace of R^2. If p is in S, let p' be the least element of S greater than p. Define the homeomorphism h taking (X, ρ) onto itself by $h(p, q) = (p', q')$ for every point (p, q) in X. The homeomorphism h is expansive on (X, ρ) with expansive constant δ for any δ less than $1/6$.

Let (Y, d) be the closure of (X, ρ) considered as a subspace of R^2, and let $\hat{h}$ be the extension of h to (Y, d) defined by

$$\begin{aligned} \hat{h}(x, 0) &= (x, 0), && x = 0, 1, \\ \hat{h}(x, 1) &= (x, 1), && x = 0, 1, \\ \hat{h}(x, p) &= (x, p'), && x = 0, 1 \text{ and } p \text{ in } S, \\ \hat{h}(p, y) &= (p', y), && y = 0, 1 \text{ and } p \text{ in } S. \end{aligned}$$

The extension $\hat{h}$ is a homeomorphism on (Y, d), and $\hat{h}$ is expansive on $Y - X$ with the expansive constant δ for δ less than $1/6$. It is also the case that h is expansive on (Y, d) for the same δ's, however, the orbit $O(1/k, 1 - 1/k)$, where $1/k < \delta$, is always within δ of $Y - X$.

EXAMPLE 2. We finally consider a simple example in which all the hypotheses of Theorem 2 are satisfied except the equivalence condition on the generators.

For n a positive integer less than or equal to nineteen, define C_n to be a subset of R^2 by

$$C_n = \left\{ (r, \theta) \,\middle|\, r = \frac{1}{n}, \theta = 0, \pi, \text{ and } \pm \frac{k\pi}{2(k+1)}, k = 1, 2, \dots, n \right\},$$

and define

$$C = \left\{ (r, \theta) \,\middle|\, r = \frac{1}{20}, \theta = 0, \pi, \text{ and } \pm \frac{k\pi}{2(k+1)}, k = 1, 2, \dots, 19 \right\}.$$

Let $Y = \{\bigcup_{n=1}^{19} C_n\} \cup C$ and obtain the space (Y, ρ) as a subspace of R^2.

Let the subspace (X, d) be given by the union of the C_n, $X = \bigcup_{n=1}^{19} C_n$. Define the homeomorphism h taking (X, d) onto itself by $h(p) = h(1/n, \theta) = (1/n, \theta')$, where $(1/n, \theta')$ is the next point counterclockwise from $(1/n, \theta)$. Then h is expansive with expansive constant $\delta = \sqrt{2}/20$.

The extension $\hat{h}$ of h defined on $Z = Y - X$ by $\hat{h}(1/20, \theta) = (1/20, \theta')$ is homeomorphic and expansive on Z with expansive constant $\delta = \sqrt{2}/20$, but h is not expansive on Y for that expansive constant.

Let $A_\alpha = \{(1/19, \alpha), (1/20, \alpha)\}$ for each α an angle for C_{19}. Then $\operatorname{diam}(A_\alpha) < \delta$ and $\mathcal{U} = \{A_\alpha \cap Z | \alpha \text{ an angle for } C_{19}\}$ is a generator for $\hat{h}$ on Z. Now define the bisequence $\{B_i\}_{i=-\infty}^{+\infty}$ of elements of $\{A_\alpha | \alpha \text{ is an angle for } C_{19}\}$ by $B_0 = A_0$, $B_i \cap Z = \hat{h}^i(B_0 \cap Z)$, i an integer. Then

$$\bigcap_{i=-\infty}^{\infty} \hat{h}^{-i}(\overline{B}_i) = \left\{ \left(\frac{1}{20}, 0 \right), \left(\frac{1}{19}, 0 \right) \right\}.$$

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