PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 1, September 1985

ON REPRESENTATIONS OF THE HOLOMORPH OF ANALYTIC GROUPS

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ABSTRACT. We completely characterize those complex analytic groups whose holomorph groups admit finite-dimensional faithful complex analytic representations.

Let G be a complex analytic group, and let $\mathrm{Aut}(G)$ denote the group of all analytic automorphisms of G, endowed with its natural structure of a complex Lie group. The semidirect product $G \circledS \mathrm{Aut}(G)$ with respect to the natural action of $\mathrm{Aut}(G)$ on G is called the holomorph of G. In [**3**], G. Hochschild has shown that if G is faithfully representable (that is, G admits a faithful finite-dimensional analytic representation) and if the maximum nilpotent normal analytic subgroup of G is simply connected, then the holomorph of G is faithfully representable. The main purpose of this paper is to give an intrinsic characterization of those complex analytic groups whose holomorphs are faithfully representable. The following is the main result:

THEOREM. *Let G be a faithfully representable complex analytic group. Then the holomorph of G is faithfully representable if and only if G satisfies one of the following*:

(i) *The maximum nilpotent normal analytic subgroup N of G is simply connected*;

(ii) $G = G'$;

(iii) G/G' *is a* 1-*dimensional complex torus.*

We note here that since G is faithfully representable, the commutator group G' is closed by a result of Goto (see [**3**, Chapter XVIII, Theorem 4.5]).

For the most part, we make use of results and techniques of earlier work by Hochschild [**2**, **3**], and also by Hochschild and Mostow on representations and representative functions of Lie groups.

Preliminary results on $R[G]$. Here we gather some of the results on representations and representative functions of Lie groups for later use. Let G be a complex analytic group. For any complex-valued function f on G and $x \in G$, define the left translate xf and the right translate fx by $(xf)(y) = f(yx)$, $(fx)(y) = f(xy)$, $y \in G$. A continuous function f: $G \to \mathbf{C}$ is called a representative function if the complex linear space spanned by the left translates xf, where x ranges over G, is finite

Received by the editors August 31, 1984.
1980 *Mathematics Subject Classification*. Primary 22E10; Secondary 22E45.
Key words and phrases. Holomorph, representative functions, reductive groups, unipotent groups.

dimensional. The representative functions on G form a $\mathbf{C}$-algebra which we denote by $R[G]$. Let ϕ be a complex analytic representation of G in a finite-dimensional linear space, and let $[\phi]$ denote the set of all complex-valued functions on G of the form $\tau \circ \phi$, where τ is a linear functional on the algebra of all linear endomorphisms of V. The set $[\phi]$ is a finite-dimensional subspace of $R[G]$ and is invariant under left and right translations by elements of G. The group $\mathrm{Aut}(G)$ acts naturally on $R[G]$ by composition $(f, \alpha) \to f \circ \alpha$ for $f \in R[G]$ and $\alpha \in \mathrm{Aut}(G)$. The $R[G]$ also admits a comultiplication γ: $R[G] \to R[G] \otimes R[G]$ with which $R[G]$ becomes a Hopf algebra. The antipode of this Hopf algebra is the map $f \to f'$, where $f'(x) = f(x^{-1})$. If $f \in R[G]$ and $x, y \in G$, then $f(xy) = \sum_j g_j(x)h_j(y)$, where $\gamma(f) = \sum_j g_j \otimes h_j$.

Suppose now G is a faithfully representable complex analytic group, and assume that G/G' is reductive. (We recall [**4**] that a complex analytic group H is called reductive if it is faithfully representable and if every finite-dimensional complex analytic representation of H is semisimple.) Then $R[G]$ is finitely generated as a $\mathbf{C}$-algebra, and G in this case is a semidirect product $G = KP$, where K is the radical of G' and P is a reductive complex analytic group (see [**5**, Theorem 5.2]). Let S be a finite-dimensional bi-stable subspace of $R[G]$ whose elements, together with the constants, generate $R[G]$, and let ϕ be the representation of G by left translations on S. Then ϕ is faithful, and $\phi(G)$ is an algebraic subgroup of the group of all linear automorphisms of S. Since every complex analytic representation of G induces a unipotent representation of K by Lie's Theorem, it follows that an analytic representation ξ of $\phi(G)$ is rational if and only if $\xi = \sigma \circ \phi^{-1}$ for some complex analytic representation σ of G. Note that the affine algebra of polynomial functions of $\phi(G)$ consists of functions of the form $f \circ \phi^{-1}$ with $f \in R[G]$. It is also clear that $\phi(K)$ is the unipotent radical of $\phi(G)$ and $\phi(P)$ is a maximal linear reductive subgroup of the algebraic group $\phi(G)$. Transporting the complex affine algebraic group structure of $\phi(G)$ to G via ϕ^{-1}, G can be endowed with the structure of a complex affine algebraic group such that the polynomial functions on G are precisely the elements of $R[G]$. In this case, K is the unipotent radical and P is a maximal linear reductive algebraic subgroup of the algebraic group G. For $\alpha \in \mathrm{Aut}(G)$, we have $f \circ \alpha \in R[G]$ whenever $f \in R[G]$. Hence, $\mathrm{Aut}(G)$ coincides with the group of all algebraic group automorphisms of the algebraic group G.

Proof of theorem. Before we prove the theorem, we first observe the following: Assume that G/G' is reductive, and let $G = KP$ be a semidirect decomposition where K is the radical of G' and P is a maximal reductive subgroup of G. By what we have discussed above, G can be given the structure of a complex affine algebraic group such that K is the unipotent radical of G. Since P is reductive, its Lie algebra is reductive in the usual sense, and this implies that the commutator subgroup P' of P is semisimple, and that $P = AP'$, where A is the connected component of 1 of the center $Z(P)$ of P. Since P' is faithfully representable, its center (and hence $A \cap P'$) is finite. Noting that $G/K \cong P$ and $G'/K \cong P'$, we see that $G/G' \cong (G/K)/(G'/K) \cong P/P' \cong A/(A \cap P')$, and hence $\dim(Z(P)) = 0$ (resp. $\dim(Z(P)) = 1$) if and only if $G = G'$ (resp. G/G' is isomorphic to the 1-dimensional complex torus).

Now we are ready to prove our theorem. Suppose that $G \circledS \mathrm{Aut}(G)$ is faithfully representable, and assume that condition (i) of the theorem does not hold. We will first show that G/G' is reductive, and that the natural action of $\mathrm{Aut}(G)$ on $R[G]$ is locally finite. (That is, each orbit $f \circ \mathrm{Aut}(G)$ spans a finite-dimensional subspace of $R[G]$.)

Let $\tilde{\rho}$ be a faithful analytic representation of $G \circledS \mathrm{Aut}(G)$, and let ρ denote the restriction of $\tilde{\rho}$ to G. For $f \in [\rho]$, choose a function $\tilde{f} \in [\tilde{\rho}]$ such that $\tilde{f}|G = f$. For any $\alpha \in \mathrm{Aut}(G)$, the translate $(1, \alpha^{-1})f(1, \alpha)$ coincides with $f \circ \alpha$ on G. Since f is a representative function on $G \circledS \mathrm{Aut}(G)$, it follows that $[\rho] \circ \mathrm{Aut}(G)$ spans a finite-dimensional linear space. Let $\mathscr{A}$ be the largest sub-Hopf algebra of $R[G]$ that is stable under the natural action of $\mathrm{Aut}(G)$ and that is locally finite as an $\mathrm{Aut}(G)$-module. The sub-Hopf algebra generated by the set $[\rho] \circ \mathrm{Aut}(G)$ is clearly contained in $\mathscr{A}$, because it is $\mathrm{Aut}(G)$-invariant and the action of $\mathrm{Aut}(G)$ on the sub-Hopf algebra is locally-finite. Since $[\rho]$ separates the points of G, $\mathscr{A}$ separates the points of G.

Next we show that G/G' is reductive. For this, we follow arguments used in the proof of Theorem 5 of [**2**] with some modification. Suppose that G/G' is not reductive. Thus there exists a closed subgroup L of G such that $G' \leqslant L$ and L/G' is the maximum torus of the abelian group G/G'. Then G/L is a (nontrivial) vector group. We can find a closed normal subgroup H of G such that $L \leqslant H$ and $G/H \cong \mathbf{C}^1$. Thus G splits over H, and we can write $G = H \cdot V$ (semidirect product), where V is a 1-dimensional vector subgroup of G. Since the nilpotent analytic group N is not simply connected, N contains a 1-dimensional central torus T. We identify V with the complex field $\mathbf{C}$, and T with the multiplicative group of nonzero complex numbers. For $c \in \mathbf{C}$, define $\alpha_c\colon G \to G$ by $\alpha_c(x) = x\exp(c\pi(x))$, where $\pi\colon G \to V$ is the projection. Clearly, the α_c's are analytic automorphisms of G. Pick a function $f \in [\rho]$ such that f is not constant on T, and choose a basis $g_1, g_2, \ldots, g_m$ for the space spanned by the right translates $f \cdot G$ of f. Then we have

$$fx = \sum_j f_j(x) g_j$$

for some $f_1, f_2, \ldots, f_m \in R[G]$. For $u \in V$, and $c \in \mathbf{C}$,

$$f \circ \alpha_c(u) = f((\exp(cu))u) = \sum_j f_j(\exp(cu)) g_j(u).$$

If z denotes the identity map on the group T, then the representative functions on T are polynomials of the form $\sum_i c_i z^i$ with $c \in \mathbf{C}$. Thus, if f_j' denotes the restriction of f_j to T, then we have $f_j' = \sum_i c_{ij} z^i$ with complex coefficients c_{ij}. Hence for each $c \in \mathbf{C}$, $f \circ \alpha_c(u) = \sum_{i,j} c_{ij} \exp(icu) g_j(u)$. Since $[\rho] \circ \mathrm{Aut}(G)$ spans a finite-dimensional space, the functions $f \circ \alpha_c$, $c \in \mathbf{C}$, all lie in a finite-dimensional space of $R[G]$, and hence we must have $c_{ij} = 0$, for $i > 0$ and for all j. This shows that the functions f_j' are all constant. But for $t \in T$, $f(t) = (ft)(1) = \sum_j f_j(t) g_j(1)$, and it follows that f is constant on T, which contradicts the choice of f. Thus, G/G' is reductive.

Now G/G' is reductive, and hence by what we have observed before, G may be viewed as a complex affine algebraic group with $R[G]$ being identified as the affine algebra of polynomial functions on G. Since the sub-Hopf algebra $\mathscr{A}$ of $R[G]$ separates the points of G, we have $\mathscr{A} = R[G]$ by [**1**, Theorem 6.6]; that is, $R[G]$ is locally finite as an Aut(G)-module. Hence, G is a conservative complex affine algebraic group in the sense of [**6**]. Let $G = KP$ be the decomposition given in the beginning of our proof. By [**6**, Theorem 3.2], either the connected component Z of 1 in the center of G is a unipotent subgroup or else the dimension of $Z(P)$ is at most 1. But since we assumed that the maximum nilpotent normal analytic subgroup N of G is not simply connected, Z cannot be a vector group, and hence Z is not unipotent. It follows that the dimension of $Z(P)$ is at most 1, showing that either $G = G'$ or G/G' is isomorphic to a 1-dimensional torus. Hence we obtain the condition (ii) or (iii).

Next we prove the converse. Suppose N is simply connected. Then G Ⓢ Aut(G) is faithfully representable by Hochschild [**3**]. Suppose now that the condition (ii) or (iii) holds. In particular, G/G' is reductive in either case, and hence G may be given a structure of an affine algebraic group with $R[G]$ being the affine algebra of polynomial functions on G. As we have observed before, the condition (ii) or (iii) is equivalent to the condition $\dim Z(P) \leqslant 1$. Hence by [**6**, Theorem 3.2], G is a conservative algebraic group, and thus the action of Aut(G) is locally-finite. Let ϕ be a finite-dimensional faithful analytic representation of G, and let W denote the linear subspace spanned by $[\phi] \circ \mathrm{Aut}(G)$. Since $[\phi]$ is finite-dimensional, so is W. Define a representation σ of G Ⓢ Aut(G) on the linear space W by $\sigma(x, \alpha)(w) = x(w \circ \alpha^{-1})$, for $(x, \alpha) \in G$ Ⓢ Aut(G), and $w \in W$. Clearly, σ is an analytic representation of G Ⓢ Aut(G). Since W separates the points of G, σ is faithful. The proof of the theorem is complete.

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