PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 1, September 1985

# ON REPRESENTATIONS OF THE HOLOMORPH OF ANALYTIC GROUPS

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ABSTRACT. We completely characterize those complex analytic groups whose holomorph groups admit finite-dimensional faithful complex analytic representations.

Let $G$ be a complex analytic group, and let $\mathrm{Aut}(G)$ denote the group of all analytic automorphisms of $G$, endowed with its natural structure of a complex Lie group. The semidirect product $G \circledS \mathrm{Aut}(G)$ with respect to the natural action of $\mathrm{Aut}(G)$ on $G$ is called the holomorph of $G$. In **[3]**, G. Hochschild has shown that if $G$ is faithfully representable (that is, $G$ admits a faithful finite-dimensional analytic representation) and if the maximum nilpotent normal analytic subgroup of $G$ is simply connected, then the holomorph of $G$ is faithfully representable. The main purpose of this paper is to give an intrinsic characterization of those complex analytic groups whose holomorphs are faithfully representable. The following is the main result:

THEOREM. *Let G be a faithfully representable complex analytic group. Then the holomorph of G is faithfully representable if and only if G satisfies one of the following*:

(i) *The maximum nilpotent normal analytic subgroup N of G is simply connected*;

(ii) $G = G'$;

(iii) $G/G'$ *is a* 1-*dimensional complex torus.*

We note here that since $G$ is faithfully representable, the commutator group $G'$ is closed by a result of Goto (see **[3]**, Chapter XVIII, Theorem 4.5]).

For the most part, we make use of results and techniques of earlier work by Hochschild **[2, 3]**, and also by Hochschild and Mostow on representations and representative functions of Lie groups.

**Preliminary results on** $R[G]$. Here we gather some of the results on representations and representative functions of Lie groups for later use. Let $G$ be a complex analytic group. For any complex-valued function $f$ on $G$ and $x \in G$, define the left translate $xf$ and the right translate $fx$ by $(xf)(y) = f(yx)$, $(fx)(y) = f(xy)$, $y \in G$. A continuous function $f$: $G \to \mathbf{C}$ is called a representative function if the complex linear space spanned by the left translates $xf$, where $x$ ranges over $G$, is finite

Received by the editors August 31, 1984.
1980 *Mathematics Subject Classification*. Primary 22E10; Secondary 22E45.
*Key words and phrases*. Holomorph, representative functions, reductive groups, unipotent groups.

dimensional. The representative functions on $G$ form a $\mathbf{C}$-algebra which we denote by $R[G]$. Let $\phi$ be a complex analytic representation of $G$ in a finite-dimensional linear space, and let $[\phi]$ denote the set of all complex-valued functions on $G$ of the form $\tau \circ \phi$, where $\tau$ is a linear functional on the algebra of all linear endomorphisms of $V$. The set $[\phi]$ is a finite-dimensional subspace of $R[G]$ and is invariant under left and right translations by elements of $G$. The group $\mathrm{Aut}(G)$ acts naturally on $R[G]$ by composition $(f, \alpha) \to f \circ \alpha$ for $f \in R[G]$ and $\alpha \in \mathrm{Aut}(G)$. The $R[G]$ also admits a comultiplication $\gamma$: $R[G] \to R[G] \otimes R[G]$ with which $R[G]$ becomes a Hopf algebra. The antipode of this Hopf algebra is the map $f \to f'$, where $f'(x) = f(x^{-1})$. If $f \in R[G]$ and $x, y \in G$, then $f(xy) = \sum_j g_j(x)h_j(y)$, where $\gamma(f) = \sum_j g_j \otimes h_j$.

Suppose now $G$ is a faithfully representable complex analytic group, and assume that $G/G'$ is reductive. (We recall [**4**] that a complex analytic group $H$ is called reductive if it is faithfully representable and if every finite-dimensional complex analytic representation of $H$ is semisimple.) Then $R[G]$ is finitely generated as a $\mathbf{C}$-algebra, and $G$ in this case is a semidirect product $G = KP$, where $K$ is the radical of $G'$ and $P$ is a reductive complex analytic group (see [**5**, Theorem 5.2]). Let $S$ be a finite-dimensional bi-stable subspace of $R[G]$ whose elements, together with the constants, generate $R[G]$, and let $\phi$ be the representation of $G$ by left translations on $S$. Then $\phi$ is faithful, and $\phi(G)$ is an algebraic subgroup of the group of all linear automorphisms of $S$. Since every complex analytic representation of $G$ induces a unipotent representation of $K$ by Lie's Theorem, it follows that an analytic representation $\xi$ of $\phi(G)$ is rational if and only if $\xi = \sigma \circ \phi^{-1}$ for some complex analytic representation $\sigma$ of $G$. Note that the affine algebra of polynomial functions of $\phi(G)$ consists of functions of the form $f \circ \phi^{-1}$ with $f \in R[G]$. It is also clear that $\phi(K)$ is the unipotent radical of $\phi(G)$ and $\phi(P)$ is a maximal linear reductive subgroup of the algebraic group $\phi(G)$. Transporting the complex affine algebraic group structure of $\phi(G)$ to $G$ via $\phi^{-1}$, $G$ can be endowed with the structure of a complex affine algebraic group such that the polynomial functions on $G$ are precisely the elements of $R[G]$. In this case, $K$ is the unipotent radical and $P$ is a maximal linear reductive algebraic subgroup of the algebraic group $G$. For $\alpha \in \mathrm{Aut}(G)$, we have $f \circ \alpha \in R[G]$ whenever $f \in R[G]$. Hence, $\mathrm{Aut}(G)$ coincides with the group of all algebraic group automorphisms of the algebraic group $G$.

**Proof of theorem.** Before we prove the theorem, we first observe the following: Assume that $G/G'$ is reductive, and let $G = KP$ be a semidirect decomposition where $K$ is the radical of $G'$ and $P$ is a maximal reductive subgroup of $G$. By what we have discussed above, $G$ can be given the structure of a complex affine algebraic group such that $K$ is the unipotent radical of $G$. Since $P$ is reductive, its Lie algebra is reductive in the usual sense, and this implies that the commutator subgroup $P'$ of $P$ is semisimple, and that $P = AP'$, where $A$ is the connected component of 1 of the center $Z(P)$ of $P$. Since $P'$ is faithfully representable, its center (and hence $A \cap P'$) is finite. Noting that $G/K \cong P$ and $G'/K \cong P'$, we see that $G/G' \cong (G/K)/(G'/K) \cong P/P' \cong A/(A \cap P')$, and hence $\dim(Z(P)) = 0$ (resp. $\dim(Z(P)) = 1$) if and only if $G = G'$ (resp. $G/G'$ is isomorphic to the 1-dimensional complex torus).

Now we are ready to prove our theorem. Suppose that $G \circledS \operatorname{Aut}(G)$ is faithfully representable, and assume that condition (i) of the theorem does not hold. We will first show that $G/G'$ is reductive, and that the natural action of $\operatorname{Aut}(G)$ on $R[G]$ is locally finite. (That is, each orbit $f \circ \operatorname{Aut}(G)$ spans a finite-dimensional subspace of $R[G]$.)

Let $\tilde{\rho}$ be a faithful analytic representation of $G \circledS \operatorname{Aut}(G)$, and let $\rho$ denote the restriction of $\tilde{\rho}$ to $G$. For $f \in [\rho]$, choose a function $\tilde{f} \in [\tilde{\rho}]$ such that $\tilde{f}|G = f$. For any $\alpha \in \operatorname{Aut}(G)$, the translate $(1, \alpha^{-1})f(1, \alpha)$ coincides with $f \circ \alpha$ on $G$. Since $f$ is a representative function on $G \circledS \operatorname{Aut}(G)$, it follows that $[\rho] \circ \operatorname{Aut}(G)$ spans a finite-dimensional linear space. Let $\mathscr{A}$ be the largest sub-Hopf algebra of $R[G]$ that is stable under the natural action of $\operatorname{Aut}(G)$ and that is locally finite as an $\operatorname{Aut}(G)$-module. The sub-Hopf algebra generated by the set $[\rho] \circ \operatorname{Aut}(G)$ is clearly contained in $\mathscr{A}$, because it is $\operatorname{Aut}(G)$-invariant and the action of $\operatorname{Aut}(G)$ on the sub-Hopf algebra is locally-finite. Since $[\rho]$ separates the points of $G$, $\mathscr{A}$ separates the points of $G$.

Next we show that $G/G'$ is reductive. For this, we follow arguments used in the proof of Theorem 5 of **[2]** with some modification. Suppose that $G/G'$ is not reductive. Thus there exists a closed subgroup $L$ of $G$ such that $G' \leqslant L$ and $L/G'$ is the maximum torus of the abelian group $G/G'$. Then $G/L$ is a (nontrivial) vector group. We can find a closed normal subgroup $H$ of $G$ such that $L \leqslant H$ and $G/H \cong \mathbf{C}^1$. Thus $G$ splits over $H$, and we can write $G = H \cdot V$ (semidirect product), where $V$ is a 1-dimensional vector subgroup of $G$. Since the nilpotent analytic group $N$ is not simply connected, $N$ contains a 1-dimensional central torus $T$. We identify $V$ with the complex field $\mathbf{C}$, and $T$ with the multiplicative group of nonzero complex numbers. For $c \in \mathbf{C}$, define $\alpha_c$: $G \to G$ by $\alpha_c(x) = x \exp(c\pi(x))$, where $\pi$: $G \to V$ is the projection. Clearly, the $\alpha_c$'s are analytic automorphisms of $G$. Pick a function $f \in [\rho]$ such that $f$ is not constant on $T$, and choose a basis $g_1, g_2, \ldots, g_m$ for the space spanned by the right translates $f \cdot G$ of $f$. Then we have

$$fx = \sum_j f_j(x) g_j$$

for some $f_1, f_2, \ldots, f_m \in R[G]$. For $u \in V$, and $c \in \mathbf{C}$,

$$f \circ \alpha_c(u) = f((\exp(cu))u) = \sum_j f_j(\exp(cu)) g_j(u).$$

If $z$ denotes the identity map on the group $T$, then the representative functions on $T$ are polynomials of the form $\sum_i c_i z^i$ with $c \in \mathbf{C}$. Thus, if $f_j'$ denotes the restriction of $f_j$ to $T$, then we have $f_j' = \sum_i c_{ij} z^i$ with complex coefficients $c_{ij}$. Hence for each $c \in \mathbf{C}$, $f \circ \alpha_c(u) = \sum_{i,j} c_{ij} \exp(icu) g_j(u)$. Since $[\rho] \circ \operatorname{Aut}(G)$ spans a finite-dimensional space, the functions $f \circ \alpha_c$, $c \in \mathbf{C}$, all lie in a finite-dimensional space of $R[G]$, and hence we must have $c_{ij} = 0$, for $i > 0$ and for all $j$. This shows that the functions $f_j'$ are all constant. But for $t \in T$, $f(t) = (ft)(1) = \sum_j f_j(t) g_j(1)$, and it follows that $f$ is constant on $T$, which contradicts the choice of $f$. Thus, $G/G'$ is reductive.

Now $G/G'$ is reductive, and hence by what we have observed before, $G$ may be viewed as a complex affine algebraic group with $R[G]$ being identified as the affine algebra of polynomial functions on $G$. Since the sub-Hopf algebra $\mathscr{A}$ of $R[G]$ separates the points of $G$, we have $\mathscr{A} = R[G]$ by [**1**, Theorem 6.6]; that is, $R[G]$ is locally finite as an $\mathrm{Aut}(G)$-module. Hence, $G$ is a conservative complex affine algebraic group in the sense of [**6**]. Let $G = KP$ be the decomposition given in the beginning of our proof. By [**6**, Theorem 3.2], either the connected component $Z$ of 1 in the center of $G$ is a unipotent subgroup or else the dimension of $Z(P)$ is at most 1. But since we assumed that the maximum nilpotent normal analytic subgroup $N$ of $G$ is not simply connected, $Z$ cannot be a vector group, and hence $Z$ is not unipotent. It follows that the dimension of $Z(P)$ is at most 1, showing that either $G = G'$ or $G/G'$ is isomorphic to a 1-dimensional torus. Hence we obtain the condition (ii) or (iii).

Next we prove the converse. Suppose $N$ is simply connected. Then $G \circledS \mathrm{Aut}(G)$ is faithfully representable by Hochschild [**3**]. Suppose now that the condition (ii) or (iii) holds. In particular, $G/G'$ is reductive in either case, and hence $G$ may be given a structure of an affine algebraic group with $R[G]$ being the affine algebra of polynomial functions on $G$. As we have observed before, the condition (ii) or (iii) is equivalent to the condition $\dim Z(P) \leqslant 1$. Hence by [**6**, Theorem 3.2], $G$ is a conservative algebraic group, and thus the action of $\mathrm{Aut}(G)$ is locally-finite. Let $\phi$ be a finite-dimensional faithful analytic representation of $G$, and let $W$ denote the linear subspace spanned by $[\phi] \circ \mathrm{Aut}(G)$. Since $[\phi]$ is finite-dimensional, so is $W$. Define a representation $\sigma$ of $G \circledS \mathrm{Aut}(G)$ on the linear space $W$ by $\sigma(x, \alpha)(w) = x(w \circ \alpha^{-1})$, for $(x, \alpha) \in G \circledS \mathrm{Aut}(G)$, and $w \in W$. Clearly, $\sigma$ is an analytic representation of $G \circledS \mathrm{Aut}(G)$. Since $W$ separates the points of $G$, $\sigma$ is faithful. The proof of the theorem is complete.

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