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NONALGEBRAIC KILLERS OF KNOT GROUPS

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ABSTRACT. We show that a knot exists with the property that there exists a killer of the knot group which is not the image of the meridian under any automorphism.

1. Introduction. Let K be a knot in the 3-sphere S^3, and let g be an element in the knot group $\pi_1(S^3 - K)$. We say the element g is a killer of the knot group if adding the relation $g = 1$ in $\pi_1(S^3 - K)$ makes the group trivial.

It is obvious that the meridian μ of the knot K is a killer of the knot group $\pi_1(S^3 - K)$, as are the conjugates of $\mu^{\pm 1}$. More generally, it is obvious that if ϕ is any automorphism of the knot group $\pi_1(S^3 - K)$, then $\phi(\mu)$ is a killer. Thus it is reasonable to ask if there is a killer of the knot group $\pi_1(S^3 - K)$ which is not the image of μ under any automorphism.

In this paper we will take a geometric approach, examples of knots with geometrically interesting killers will be given, and these killers are shown not to be images of μ under any automorphism.

I would like to thank the referee for pointing out the necessity of Lemma 3.10 for the hypothesis of Theorem 3.11. The proof of this lemma is also due to him.

2. Preliminaries. In this paper, we work in the piecewise linear category. Let M and N be manifolds, and let $f\colon M \to N$ be a map; the induced homomorphisms, $\pi_1(M) \to \pi_1(N)$ or $H_1(M) \to H_1(N)$, are both denoted by f_*. In particular, if M is a submanifold of N, then i_* is the inclusion-induced homomorphism. A knot is a piecewise linear homeomorphic image of the circle S^1 in the 3-sphere S^3. Homeomorphism of spaces, isomorphism of groups, homotopy of maps, homological equivalence are denoted by $\cong$, $\approx$, $\simeq$, $\sim$, respectively. Let M be a manifold; the interior and boundary of M are denoted $\operatorname{Int}(M)$, $\operatorname{Bdy}(M)$, respectively. A regular neighborhood of a knot K in S^3 is denoted $N(K)$, and the closure of the complement of $N(K)$ is denoted $C^3(K)$. Note that $S^3 - N(K) \cong S^3 - K$, and both are homotopy equivalent to $C^3(K)$. When choice of base point is not important, the groups $\pi_1(S^3 - K)$, $\pi_1(C^3(K))$, and $\pi_1(S^3 - N(K))$ are viewed as identical. There are, up to isotopy, unique simple closed curves μ and λ on $\operatorname{Bdy}(C^3(K))$ such that μ bounds a disk in $N(K)$, and $\lambda \sim 0$ in $C^3(K)$. A choice of the pair (μ, λ), with any fixed orientation, is called a preferred meridian and longitude pair for K, $C^3(K)$, and $N(K)$.

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A killer g of a knot group G is *nonalgebraic* if g is not the image of μ under any automorphism G.

Let V be a solid torus; a curve ξ contained in $\mathrm{Int}(V)$ is *geometrically essential* in V if there is no 3-ball D^3 such that $\xi \subset D^3 \subset V$. Let V' be a standardly embedded solid torus in S^3, and let K' be a geometrically essential simple closed curve in V' that is not a core of V' (i.e. V' cannot be realized as $K' \times D^2$). Let $K_1 \subset S^3$ be another knot and let $V = N(K_1)$ be a tubular neighborhood of K_1 in S^3. Finally, let h: $V' \to V$ be a homeomorphism and let K denote $h(K')$. We say K is a satellite knot about the knot K_1, and call K_1 a companion of K_2. (Note that we are not requiring that h be faithful, as is usually done for these definitions.) Let V be a solid torus and let $K \subset V$ be a knot. The wrapping number of K in V, denoted $\mathrm{wr}(V, K)$, is the minimum number of points of intersection of K with any meridional disk of V. The winding number of K in V, denoted $\mathrm{wi}(V, K)$, is the absolute value of the homology class of K in $H_1(V; Z) \approx Z$.

Let X be a 2-manifold and let M be a compact 3-manifold containing a 2-sided surface Y. A map f: $X \to M$ is transverse with respect to Y if there exist product neighborhoods $Y \times [1,1]$ of $Y = Y \times 0$ in M and $f^{-1}(Y) \times [-1,1]$ of $f^{-1}(Y) = f^{-1}(Y) \times 0$ in X such that f maps each fiber $x \times [-1,1]$ homeomorphically to the fiber $f(x) \times [-1,1]$ for each $x \in f^{-1}(Y)$.

The following definitions and proposition are adapted from [**5**].

Let M be an orientable 3-manifold, and let $T \subset \mathrm{Bdy}(M)$ be compact and incompressible. The 3-manifold pair (M, T) is called acceptable if $\pi_2(M) = 1$. By this definition, and the asphericity of knot manifolds for any knot K, the pair $(C^3(K), \mathrm{Bdy}(C^3(K)))$ is acceptable. An orientable 3-manifold is called exceptional if M is compact and if each component of $\mathrm{Bdy}(M)$ is a torus T such that $\mathrm{im}(\pi_1(T) \to \pi_1(M))$ has index $\leqslant 2$ in $\pi_1(M)$. By this definition it is clear that no knot manifold is exceptional, since $\pi_1(\mathrm{Bdy}(C^3(K)))$ would have to be normal in $\pi_1(C^3(K))$. The following proposition is from [**5**, Corollary 1, p. 321].

PROPOSITION 2.1. *Let (M, T) be an acceptable 3-manifold pair such that T is a torus but M is not exceptional. Then any two singular loops in T which are homotopic in M are either homotopic or antihomotopic in T.*

3. Nonalgebraic killers of knot groups. The examples of knots with nonalgebraic killers that we will exhibit will be satellite knots $h(K')$, where h is a homeomorphism and K' is a knot contained in a standardly embedded solid torus V' in S^3; the nonalgebraic killers will be the meridian of $h(V')$.

We first obtain some restrictions on such killers by homology considerations as follows: Let $K \subset \mathrm{Int}(V)$ be a satellite knot contained in the interior of a knotted solid torus V in S^3, and let $T = \mathrm{Bdy}(V)$. Suppose (μ_1, λ_1) is a preferred meridian and longitude pair of V in T. Let $N(K)$ be a regular neighborhood of K in $\mathrm{Int}(V)$, and suppose (μ, λ) is a meridian and longitude pair for K on $\mathrm{Bdy}(N(K))$. Let w denote $\mathrm{wi}(V, K)$. For any loop ξ on T, the homology class in T of ξ can be represented as $\xi = m\mu_1 + n\lambda_1$, $m, n \in Z$. Since $\mu_1 \sim w\mu$ in $S^3 - K$ and since $\lambda_1 \sim 0$

in $S^3 - V$, hence in $S^3 - K$, it follows that $\xi \sim m\mu_1 + n\lambda_1 \sim m(w\mu) = mw\mu$ in $S^3 - K$. We then have the following

LEMMA 3.1. *The loop ξ is a killer for $H_1(S^3 - K)$ if and only if $mw = 1$ or -1.*

PROOF. This follows from $H_1(S^3 - K)/\langle \xi \rangle = \langle \mu \rangle / \langle mw\mu \rangle = Z/\langle mw \rangle = Z_{mw}$ and $Z_{mw} = 0$ if and only if $mw = \pm 1$.

Since a homotopy killer of $S^3 - K$, i.e. a killer of the knot group $\pi_1(S^3 - K)$ must be a homology killer of $S^3 - K$, Lemma 3.1 implies that the only possible killers of $\pi_1(S^3 - K)$ on T must be of the form $\mu_1^{\pm 1}\lambda_1^n$, $n \in Z$ and $w = \mathrm{wi}(V, K) = \pm 1$.

We now construct an example of a satellite knot in a knotted solid torus in S^3, for which the meridian of the solid torus is a killer of the knot group $\pi_1(S^3 - K)$ of K. On the other hand, we will show that no such curve can be freely homotopic to the meridian of K, and each automorphism of one of the knots we construct must send the meridian μ to a conjugate of $\mu^{\pm 1}$. It will then follow that such killers are nonalgebraic.

The construction may be simplified and reduced to the construction of a knot in an unknotted solid torus in S^3 by the following observation:

LEMMA 3.2. *Let K be a satellite knot defined by a homeomorphism h: $V' \to V$ such that $h(K') = K$, where V' is a standardly embedded solid torus in S^3, and $K' \subset V'$ is geometrically essential. Let (μ_1', λ_1') and (μ_1, λ_1) be preferred meridian and longitude pairs for V' and V respectively. Then we have*

$$\pi_1(S^3 - K)/\langle \mu_1 \rangle \approx \pi_1(V' - K')/\langle \mu_1', \lambda_1' \rangle.$$

PROOF. Since $h(V') = V$ and $h(K') = K$, we have $h(V' - K) = V - K$ and therefore the induced map h_*: $\pi_1(V' - K') \to \pi_1(V - K)$ is an isomorphism. Since h maps the solid torus V' homeomorphically onto V, we have $h_*(\mu_1') = \mu_1^{\pm 1}$, and $h_*(\lambda_1') = \mu_1^n\lambda_1^{\pm 1}$ for some $n \in Z$. Therefore via h_*, we have

$$\begin{aligned}\pi_1(V' - K')/\langle \mu_1', \lambda_1' \rangle &\approx \pi_1(V - K)/\langle \mu_1, \mu^n\lambda_1^{\pm 1} \rangle \\ &= \pi_1(V - K)/\langle \mu_1, \lambda_1^{\pm 1} \rangle.\end{aligned}$$

On the other hand, since V is knotted and K is geometrically essential in V, it follows from the Loop Theorem and Dehn's Lemma that $\mathrm{Bdy}(V)$ is incompressible in $S^3 - \mathrm{Int}(V)$ and in $V - K$. Thus by the Van Kampen Theorem, we have the amalgamated free product

$$\pi_1(S^3 - K) = \pi_1(S^3 - V) \underset{\pi_1(T)}{*} \pi_1(V - K)$$

where $\pi_1(T) = \langle \mu_1 \rangle + \langle \lambda_1 \rangle$. So since μ_1 kills all of $\pi_1(S^3 - V)$, we have

$$\pi_1(S^3 - K)/\langle \mu_1 \rangle = \pi_1(V - K)/\langle \mu_1, \lambda_1 \rangle.$$

Therefore, $\pi_1(S^3 - K)/\langle \mu_1 \rangle = \pi_1(V' - K')/\langle \mu_1', \lambda_1' \rangle$.

By the above observation, we may reduce our search for a satellite knot K in a knotted solid torus in S^3, to the search for a knot K' lying in a standardly embedded unknotted torus V' in S^3 satisfying $\pi_1(V' - K')/\langle \mu_1', \lambda_1' \rangle = 1$.

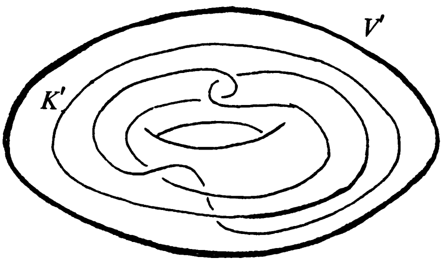

FIGURE 1

Let V' be a standardly embedded solid torus in S^3 and let K' be the knot contained in V' as shown in Figure 1. Notice that the knot K' is actually the trefoil knot in S^3, and it is put in V' so that $\mathrm{wi}(V', K') = 1$ and $\mathrm{wr}(V', K') = 3$.

LEMMA 3.3. *Let K' and V' be the knot and the solid torus as in Figure* 1, *and let (μ_1', λ_1') be the meridian and longitude pair for V'. Then $\pi_1(V' - K')/\langle \mu_1', \lambda_1' \rangle = 1$.*

PROOF. Let $K' \cap L'$ be the link shown in Figure 2. The inclusion $i\colon V' - K' \to S^3 - (K' \cup L')$ is a homotopy equivalence; adding the relation $\lambda_1' = 1$ to $\pi_1(V' - K')$ corresponds (by Van Kampen's Theorem) to filling in the component L' of the link $K' \cup L'$. Thus we have

$$\pi_1(V' - K')/\langle \mu_1', \lambda_1' \rangle \approx \pi_1(S^3 - K')/\langle \mu_1' \rangle.$$

Therefore it is sufficient to show $\pi_1(S^3 - K')/\langle \mu_1' \rangle$ is trivial.

From Figure 2, we may find the presentation of $\pi_1(S^3 - K')$ as follows:

$$\begin{aligned}\pi_1(S^3 - K') &= \langle a, b, c\colon ba = cb,\, c^{-1}bcb = bc,\, cb = ac \rangle \\ &= \langle a, b, c\colon ba = cb,\, bcb = cbc,\, cb = ac \rangle\end{aligned}$$

and $\mu_1' = a^{-2}c$. Hence,

$$\pi_1(S^3 - K')/\langle \mu_1' \rangle = \langle a, b, c\colon ba = cb,\, c^{-1}bcb = bc,\, cb = ac,\, a^{-2}c = 1 \rangle.$$

By $a^{-2}c = 1$, we get $c = a^2$. Substituting in $cb = ac$ we obtain $a^2b = aa^2$ or $b = a$. Substituting $b = a$ and $c = a^2$ in $ba = cb$ we obtain $aa = a^2a$, so $a = 1$, and therefore $b = c = 1$. Hence, we have shown $\pi_1(S^3 - K')/\langle \mu_1' \rangle = 1$, therefore $\pi_1(V' - K')/\langle \mu_1', \lambda_1' \rangle = 1$.

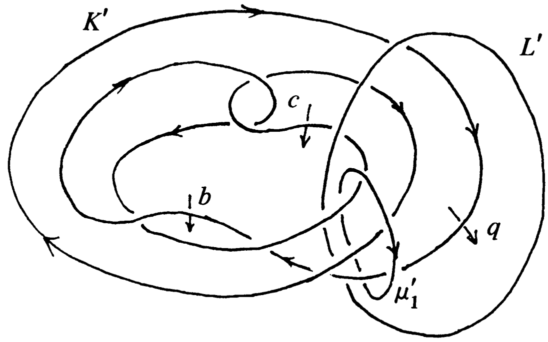

FIGURE 2

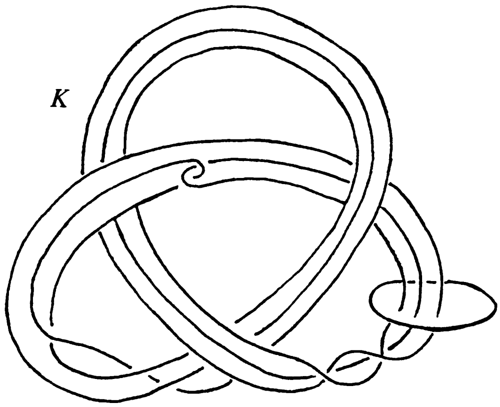

FIGURE 3

THEOREM 3.4. *For the satellite knot K in Figure* 3, *or any other knot obtained from K' by a nontrivial embedding of the solid torus V' of Figure* 1 *into S^3, the loop μ_1 is a killer for the group of the knot K.*

PROOF. This is a direct application of Lemmas 3.2 and 3.3.

Next we shall show that the killer μ_1 of the knot group constructed as above is not a conjugate of $\mu^{\pm 1}$. Instead of proving it algebraically, we will give a geometric proof. First we note that if K is a knot in S^3 with meridian μ, and if $*$ is a fixed base point in $S^3 - K$, then every element g in $\pi_1(S^3 - K, *)$ can be represented by a loop γ in $S^3 - K$ based at $*$, i.e. g is the homotopy class $[\gamma]$. Therefore, g is a conjugate of $\mu^{\pm 1}$ in $\pi_1(S^3 - K, *)$ if and only if γ is freely homotopic to $\mu^{\pm 1}$ in $S^3 - K$. Thus we may turn the question whether a killer g of the knot group $\pi_1(S^3 - K, *)$ must be conjugate to $\mu^{\pm 1}$ into the question whether a loop γ representing such an element g must be freely homotopic to $\mu^{\pm 1}$ in $S^3 - K$. Therefore, it is sufficient to show that μ_1, as a loop in $S^3 - K$, is not freely homotopic to the meridian μ of K.

The following lemma is well known and we omit its proof.

LEMMA 3.5. *Let $K \subset T^3$ be a geometrically essential curve in a solid torus T^3, and let $N(K)$ be a regular neighborhood of K in T^3. Then $T^3 - \operatorname{Int}(N(K))$ is boundary irreducible.*

THEOREM 3.6. *Let T^3 be a solid torus in S^3 and let $K \subset T^3$ be a geometrically essential curve with $\operatorname{wr}(T^3, K) \geqslant 2$. If μ_1 is a meridian of T^3, and μ is a meridian of $N(K)$, then μ_1 is not freely homotopic to μ in $T^3 - \operatorname{Int}(N(K))$.*

PROOF. Suppose μ_1 is freely homotopic to μ in $T^3 - \operatorname{Int}(N(K))$ and let

$$H\colon \big(S^1 \times I, \operatorname{Bdy}(S^1 \times I)\big) \to \big(T^3 - \operatorname{Int}(N(K)), \operatorname{Bdy}(T^3 - \operatorname{Int}(N(K)))\big)$$

be the homotopy between μ_1 and μ such that $H(S^1 \times 0) = \mu_1$, and $H(S^1 \times 1) = \mu$. We claim H is an essential map of the annulus $S^1 \times I$.

Take a base point $s \in S^1 \times 1$ in $S^1 \times I$ and a base point $x = H(s)$ in $T^3 - \mathrm{Int}(N(K))$. Then for any loop ξ based at s in $S^1 \times I$, $\xi \simeq (S^1 \times 1)^m$ for some $m \in Z$, so if $H(\xi) \simeq 1$ in $T^3 - \mathrm{Int}(N(K))$, then $\mu^m \simeq 1$ in T^3. This implies $m = 0$ since μ is a meridian of $N(K)$ and $\mathrm{Bdy}(N(K))$ is incompressible in $T^3 - \mathrm{Int}(N(K))$, and so H_* is monic.

Let α be a nonseparating arc in $S^1 \times I$. Then the components of $\mathrm{Bdy}(f(\alpha))$ are in different components of $\mathrm{Bdy}(T^3 - \mathrm{Int}(N(K)))$, so $f(\alpha)$ cannot be homotopic relative $\mathrm{Bdy}(f(\alpha))$ to an arc in $\mathrm{Bdy}(T^3 - \mathrm{Int}(N(K)))$. Hence H is essential. By the Annulus Theorem [**2**], there is an essential embedding $E\colon (S^1 \times I, \mathrm{Bdy}(S^1 \times I)) \to (T^3 - \mathrm{Int}(N(K)), \mathrm{Bdy}(T^3 - \mathrm{Int}(N(K))))$ such that $E(\mathrm{Bdy}(S^1 \times I)) = \mu_1 \cup \mu$, i.e. there is a real annulus A in $T^3 - \mathrm{Int}(N(K))$ such that $\mathrm{Bdy}(A) = \mu_1 \cup \mu$.

Since μ is a meridian of $N(K)$, there is meridional disk D in $N(K)$ with $\mathrm{Bdy}(D) = \mu$. Then $A \cup D$ is a meridional disk of T^3 with $\mathrm{Bdy}(A \cup D) = \mu_1$. But then we would have $\mathrm{wr}(T^3, K) = 1$, which contradicts our assumption that the wrapping number is $\geqslant 2$.

THEOREM 3.7. *Let T^3 be a knotted solid torus in S^3, and let $K \subset T^3$ be a knot geometrically essential in T^3. If μ_1 is a meridian of T^3 and ξ is any loop in* $\mathrm{Bdy}(T^3)$, *we have* (1) *if $\mu_1 \simeq \xi$ freely in $T^3 - \mathrm{Int}(N(K))$, then $\mu_1 \simeq \xi$ freely in* $\mathrm{Bdy}(T^3)$, (2) *if $\mu_1 \simeq \xi$ freely in $S^3 - \mathrm{Int}(T^3)$, then $\mu_1 \simeq \xi$ freely in* $\mathrm{Bdy}(T^3)$.

PROOF. (1) Since $\mu_1 \sim 0$ in T^3, so $\xi \sim 0$ in T^3. Hence $\xi \sim \mu_1^n$ in $\mathrm{Bdy}(T^3)$ for some $n \in Z$. By Lemma 3.2, $\mathrm{Bdy}(T^3)$ is incompressible in $T^3 - \mathrm{Int}(N(K))$, and therefore we can apply [**5**, Proposition 1, p. 299] to conclude that ξ and μ_1 have the same divisibility in $\mathrm{Bdy}(T^3)$. Since μ_1 is primary in $\pi_1(\mathrm{Bdy}(T^3))$, $n = 1$. Hence $\xi \simeq \mu_1$ freely in $\mathrm{Bdy}(T^3)$.

(2) We have seen that the pair $(S^3 - \mathrm{Int}(T^3), \mathrm{Bdy}(T^3))$ is acceptable, and, as a knot manifold, $S^3 - \mathrm{Int}(T^3)$ is not exceptional, so by Proposition 2.1, either $\xi \simeq \mu_1$ freely in $\mathrm{Bdy}(T^3)$ or $\xi \simeq \mu_1^{-1}$ freely in $\mathrm{Bdy}(T^3)$.

If $\xi \simeq \mu_1^{-1}$ in $\mathrm{Bdy}(T^3)$, then $\xi \simeq \mu_1^{-1}$ freely in $S^3 - T^3$; also since $\xi \simeq \mu_1$ freely in $S^3 - T^3$ by assumption, we have $\mu_1 \simeq \mu_1^{-1}$ freely in $S^3 - T^3$. By considering homology, we obtain $\mu_1 = \mu_1^{-1}$ in $H_1(S^3 - T^3) = \langle \mu_1 \rangle$, which is impossible. Hence we must have $\xi \simeq \mu_1$ freely in $\mathrm{Bdy}(T^3)$, and the proof is completed.

LEMMA 3.8. *Let X be a 2-manifold and let M be a compact 3-manifold. Let F: $X \to M$ be a map transverse with respect to a 2-sided surface Y in M. If there is a 2-disk D in X with $F(D) \cap Y \times [-1, 0) = \varnothing$ and $F|_{\mathrm{Bdy}(D)}$: $\mathrm{Bdy}(D) \to Y$ is null-homotopic, then there is a map F': $X \to M$ transverse with respect to Y and 2-disk D' in X such that* (1) $D \subset \mathrm{Int}(D')$, (2) $F'(D') \cap Y = \varnothing$, (3) $F|_{X - D'} = F^*|_{X - D'}$.

PROOF. Since $\mathrm{Bdy}(D) \subset F^{-1}(Y)$, and since F is transverse with respect to Y, for $x \in \mathrm{Bdy}(D)$, $F(x, t) = (F(x), t)$ for every $t \in [-1, 1]$. Also, since $F(D) \cap Y \times [-1, 0) = \varnothing$, thus $D \cap \mathrm{Bdy}(D) \times [-1, 0) = \varnothing$. Therefore, $D' = D \cup (\mathrm{Bdy}(D) \times [-1, 0])$ is a 2-disk in X with $D \subset \mathrm{Int}(D')$.

Now since $F(\mathrm{Bdy}(D')) \subset Y \times -1$ and $F|_{\mathrm{Bdy}(D')} \simeq F|_{\mathrm{Bdy}(D)}$ in $Y \times [-1, 0]$, $F|_{\mathrm{Bdy}(D)} \simeq 1$ in $Y \times 0$, hence $F|_{\mathrm{Bdy}(D')} \simeq 1$ in $Y \times [-1, 0]$; therefore, $F|_{\mathrm{Bdy}(D')} \simeq 1$ in

$Y \times -1$. Now extend $F|_{\mathrm{Bdy}(D')}$: $\mathrm{Bdy}(D') \to Y \times -1$ to a map F': $D' \to Y \times -1$. It is easy to see that F' and D' constructed as above satisfy (1)–(3).

THEOREM 3.9. *Let T^3 be a knotted solid torus in S^3, and let $K \subset T^3$ be a knot geometrically essential in T^3 with* $\mathrm{wr}(T^3, K) \geqslant 2$. *If μ_1 is a meridian of T^3 and μ is a meridian of $N(K) \subset T^3$, then μ_1 is not freely homotopic to μ in $S^3 - \mathrm{Int}(N(K))$.*

PROOF. Suppose to the contrary that μ_1 is freely homotopic to μ in $S^3 - \mathrm{Int}(N(K))$, and let F: $S^1 \times I \to S^3 - \mathrm{Int}(N(K))$ be a homotopy between μ_1 and μ such that $F(S^1 \times 0) = \mu_1$ and $F(S^1 \times 1) = \mu$.

Push μ_1 a little inside of $T^3 - \mathrm{Int}(N(K))$ to get μ_1' and a homotopy F' between μ_1' and μ. Put F' in general position with respect to $\mathrm{Bdy}(T^3)$ in $S^3 - \mathrm{Int}(N(K))$. Since $\mu_1' \cap \mathrm{Bdy}(T^3) = \varnothing$, we may hold $F'|_{S^1 \times 0}$ fixed, and then rebuild a homotopy between μ_1 and μ by using an annulus normal to $\mathrm{Bdy}(T^3)$ to get from μ_1 to μ_1' and then continuing via F'. Hence we may assume, without loss of generality, that the original homotopy F was obtained by such modifications, so F is transverse with respect to $\mathrm{Bdy}(T^3)$. Therefore, $F^{-1}(\mathrm{Bdy}(T^3))$ consists of $S^1 \times 0$ and a finite number of simple closed curves in $\mathrm{Int}(S^1 \times I)$. We choose such an F with the minimum number of simple closed curves in $F^{-1}(\mathrm{Bdy}(T^3))$.

We claim that there are no simple closed curves in $F^{-1}(\mathrm{Bdy}(T^3))$ that bound disks in $S^1 \times I$.

Suppose there is a simple closed curve J in $F^{-1}(\mathrm{Bdy}(T^3))$ which bounds a disk in $S^1 \times I$. Choose J to be innermost, that is, J bounds a disk D in $S^1 \times I$ with no curves of $F^{-1}(\mathrm{Bdy}(T^3))$ contained in $\mathrm{Int}(D)$. Since F is in general position with respect to $\mathrm{Bdy}(T^3)$, and $\mathrm{Bdy}(T^3)$ is a separating surface in $S^3 - \mathrm{Int}(N(K))$, so $F(D)$ is either contained in $T^3 - \mathrm{Int}(N(K))$ or contained in $S^3 - \mathrm{Int}(T^3)$. Assume that $F(D) \subset T^3 - \mathrm{Int}(N(K))$. By Lemma 3.8, there is a map F': $S^1 \times I \to S^3 - \mathrm{Int}(N(K))$ transverse with respect to $\mathrm{Bdy}(T^3)$, and there is a disk D' in $S^1 \times I$ such that $\mathrm{Int}(D') \subset D$, $F'(D') \cap \mathrm{Bdy}(T^3) = \varnothing$ and $F'|_{X - D'} = F|_{X - D'}$. Therefore, $F'^{-1}(\mathrm{Bdy}(T^3))$ has fewer components than $F^{-1}(\mathrm{Bdy}(T^3))$, which contradicts the minimality of F. Similarly, if $F(D) \subset S^3 - \mathrm{Int}(T^3)$, we can reduce the number of components of $F^{-1}(T^3)$, which also contradicts the minimality of F.

We now have that $F^{-1}(\mathrm{Bdy}(T^3))$ is a finite number of simple closed curves in $S^1 \times I$ parallel to $S^1 \times 0$. Order these nontrivial simple closed curves of $F^{-1}(\mathrm{Bdy}(T^3))$ from the bottom $S^1 \times 0$ to the top $S^1 \times 1$, and denote them by $\alpha_1, \alpha_2, \ldots, \alpha_n$. By the construction of F, we have $\mu_1 \simeq F(\alpha_1)$ freely in $T^3 - \mathrm{Int}(N(K))$, hence $\mu_1 \simeq F(\alpha_1)$ freely in $\mathrm{Bdy}(T^3)$ by Theorem 3.7(1). Since $F(\alpha_1) \simeq F(\alpha_2)$ freely in $S^3 - \mathrm{Int}(T^3)$, so $\mu_1 \simeq F(\alpha_2)$ freely in $S^3 - \mathrm{Int}(T^3)$. By Theorem 3.7(2), $\mu_1 \simeq F(\alpha_2)$ in $\mathrm{Bdy}(T^3)$. Continuing in this way, we finally obtain $\mu_1 \simeq F(\alpha_n)$ freely in $\mathrm{Bdy}(T^3)$. Now, since $F(\alpha_n) \simeq \mu$ freely in $T^3 - \mathrm{Int}(N(K))$, we get $\mu_1 \simeq \mu$ freely in $T^3 - \mathrm{Int}(N(K))$. This contradicts Theorem 3.6.

The following lemma and its proof is due to the referee:

LEMMA 3.10. *Let (V', K') be the solid torus and knot of Figure* 1, *let K be a nontrivial knot, and let h: $V' \to N(K)$ be a homeomorphism. Then $h(K')$ is not a cable knot.*

PROOF. If $h(K')$ were a cable knot, then there would exist a knot $\bar{K}$ with tubular neighborhood $N(\bar{K})$ such that $h(K') \subset \operatorname{Int} N(\bar{K})$ and $\operatorname{wr}(N(\bar{K}), h(K')) = \operatorname{wi}(N(\bar{K}), h(K')) \geqslant 2$. Since $\bar{K}$ is the *unique maximal* companion of $h(K')$ [**4**, Theorem 3, p. 250], it follows that K must be a companion of $\bar{K}$, and we can assume that $N(\bar{K}) \subset \operatorname{Int} N(K)$ [**4**, p. 250].

Now K is actually a proper companion of $\bar{K}$, that is, $K \neq \bar{K}$, since $\operatorname{wr}(N(K), h(K')) = 3$ and $\operatorname{wi}(N(K), h(K')) = 1$. Therefore $\partial N(\bar{K})$ is an essential (incompressible and not boundary parallel) torus in $N(K) - \operatorname{Int} N(h(K'))$, and so $h^{-1}(\partial N(\bar{K}))$ is essential in $C^3(K')$, because a core of $h^{-1}(N(\bar{K}))$ is not a core of V', which is a trivial knot. Since K' is simple, $C^3(K')$ contains no essential tori, and we have a contradiction. This completes the proof.

THEOREM 3.11. *For any satellite knot K obtained from K' by a nontrivial embedding of the solid torus V' of Figure* 1 *into S^3, the loop μ_1 is a nonalgebraic killer for the group of the knot K.*

PROOF. By Theorem 3.9, μ_1 is not freely homotopic to the meridian μ of K in $C^3(K)$, hence μ_1 is not a conjugate of μ in $\pi_1(C^3(K))$. Therefore, it is sufficient to prove that every automorphism of $\pi_1(C^3(X))$ sends μ to a conjugate of $\mu^{\pm 1}$.

We consider two cases. First, if $C^3(K)$ contains a properly embedded essential annulus, then K is either a composite knot, a torus knot, or a cable knot. By Lemma 3.10, the knot K is not a cable knot. If K is a composite knot, then by Theorem 3.1 of [**6**], every automorphism of the knot group of K takes μ to a conjugate of $\mu^{\pm 1}$. The knot K cannot be a torus knot; since K is a satellite knot, hence it is not a simple knot, but every torus knot is simple [**4**].

If $C^3(K)$ contains no properly embedded annuli, then by [**3**] any automorphism of the knot group $\pi_1(C^3(K))$ is induced by an autohomeomorphism of the knot manifold $C^3(K)$. Since K' is the trefoil knot in S^3, K' has Property P, so K has Property P [**1**]. Therefore the inducing homeomorphism sends μ to $\mu^{\pm 1}$, hence the automorphism takes μ to a conjugate of $\mu^{\pm 1}$. The proof is completed.

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