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A q-POLYNOMIAL IDENTITY

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ABSTRACT. We show that the q-polynomial coefficients $\left[{n \cdot m \atop r}\right]$ (see the definition preceding Theorem 1), which are the generating functions of the number of inversions between multisets, satisfy the beautiful identity

$$\prod_{i=1}^{n}\left(\sum_{j=0}^{m} q^{jm(i-1)+\binom{j}{2}} t^j\right) = \sum_{r=0}^{nm}\begin{bmatrix} n \cdot m \\ r \end{bmatrix} q^{\binom{r}{2}} t^r.$$

1. Inversions between multisets. Let $(m_1, \ldots, m_n)$ denote the *multiset* on $\{1, \ldots, n\}$, in which the multiplicity of i is m_i. The number of elements in $(m_1, \ldots, m_n)$ is $m_1 + \cdots + m_n$ and is denoted by $|(m_1, \ldots, m_n)|$. We abbreviate the multiset $(m_1, \ldots, m_n)$, where $m_1 = \cdots = m_n = m$, to $(n \cdot m)$.

A *multisubset* $(a_1, \ldots, a_n)$ of $(n \cdot m)$ satisfies $a_i \leq m$, for $i = 1, \ldots, n$, and it uniquely defines a *complementary multisubset* $(\bar{a}_1, \ldots, \bar{a}_n)$ satisfying $a_i + \bar{a}_i = m$ $(i = 1, \ldots, n)$.

An *inversion* between the multisets $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, in that order, is a pair (i, j), where i is an element of $(a_1, \ldots, a_n)$, and j is an element of $(b_1, \ldots, b_n)$, and $i > j$.

Let $(a_1, \ldots, a_n)$ be a multisubset of $(n \cdot m)$. We denote by $I(a_1, \ldots, a_n)$ the number of inversions between $(a_1, \ldots, a_n)$ and $(\bar{a}_1, \ldots, \bar{a}_n)$. We let $\left[{n \cdot m \atop r}\right]$ denote the generating function

$$\begin{bmatrix} n \cdot m \\ r \end{bmatrix} = \sum_{|(a_1, \ldots, a_n)| = r} q^{I(a_1, \ldots, a_n)}.$$

Our first theorem is a generalization of both the Gaussian recurrence for the q-binomial (Gaussian) coefficients [**2**, p. 100] and the polynomial coefficient recurrence [**1**, p. 77].

THEOREM 1 [**3**].

$$\begin{bmatrix} n \cdot m \\ r \end{bmatrix} = \sum_{i=0}^{m} q^{i((n-1)m-(r-i))} \begin{bmatrix} (n-1) \cdot m \\ r-i \end{bmatrix}.$$

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PROOF.

$$\begin{aligned}\begin{bmatrix} n\cdot m \\ r \end{bmatrix} &= \sum_{|(a_1,\ldots,a_n)|=r} q^{I(a_1,\ldots,a_n)} \\ &= \sum_{i=0}^{m} \sum_{|(a_1,\ldots,a_{n-1},i)|=r} q^{I(a_1,\ldots,a_{n-1},i)} \\ &= \sum_{i=0}^{m} q^{i(\bar{a}_1+\cdots+\bar{a}_{n-1})} \sum_{|(a_1,\ldots,a_{n-1})|=r-i} q^{I(a_1,\ldots,a_{n-1})} \\ &= \sum_{i=0}^{m} q^{i((n-1)m-(r-i))} \begin{bmatrix} (n-1)\cdot m \\ r-i \end{bmatrix},\end{aligned}$$

where the third equality is explained by

$$\begin{aligned}&I(a_1,a_2,\ldots,a_{n-1},i) \\ &\quad = a_2\bar{a}_1 + a_3(\bar{a}_1+\bar{a}_2) + \cdots + a_{n-1}(\bar{a}_1+\cdots+\bar{a}_{n-2}) + i(\bar{a}_i+\cdots+\bar{a}_{n-1}) \\ &\quad = I(a_1,\ldots,a_{n-1}) + i(\bar{a}_1+\cdots+\bar{a}_{n-1}). \quad \text{Q.E.D.}\end{aligned}$$

Because of Theorem 1, we will call $\begin{bmatrix} n\cdot m \\ r \end{bmatrix}$, a *q-polynomial coefficient.*

2. Identity.

THEOREM 2.

$$\prod_{i=1}^{n} \left(\sum_{j=0}^{m} q^{jm(i-1)+\binom{j}{2}} t^j \right) = \sum_{r=0}^{nm} \begin{bmatrix} n\cdot m \\ r \end{bmatrix} q^{\binom{r}{2}} t^r.$$

PROOF. We prove the identity by induction on n, using Theorem 1. The identity obviously holds when $n = 1$. Assume that it holds for $n-1$. Then,

$$\begin{aligned}&\prod_{i=0}^{n} \left(\sum_{j=0}^{m} q^{jm(i-1)+\binom{j}{2}} t^j \right) \\ &\quad = \left(\prod_{i=0}^{n-1} \left(\sum_{j=0}^{m} q^{jm(i-1)+\binom{j}{2}} t^j \right) \right) \left(\sum_{j=0}^{m} q^{jm(n-1)+\binom{j}{2}} t^j \right) \\ &\quad = \left(\sum_{k=0}^{(n-1)m} \begin{bmatrix} (n-1)\cdot m \\ k \end{bmatrix} q^{\binom{k}{2}} t^k \right) \left(\sum_{j=0}^{m} q^{jm(n-1)+\binom{j}{2}} t^j \right) \\ &\quad = \sum_{r=0}^{nm} \left(\sum_{i=0}^{m} \begin{bmatrix} (n-1)\cdot m \\ r-i \end{bmatrix} q^{i((n-1)m-(r-i))} \right) q^{\binom{r}{2}} t^r \\ &\quad = \sum_{r=0}^{nm} \begin{bmatrix} n\cdot m \\ r \end{bmatrix} q^{\binom{r}{2}} t^r. \quad \text{Q.E.D.}\end{aligned}$$

If we set $m = 1$ in Theorem 2, we obtain the well-known identity [**2**, p. 101]

$$(1+t)(1+tq)\cdots(1+tq^{n-1}) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{\binom{r}{2}} t^r.$$

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