A q-POLYNOMIAL IDENTITY

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ABSTRACT. We show that the q-polynomial coefficients \( \begin{bmatrix} n \cdot m \end{bmatrix} \) (see the definition preceding Theorem 1), which are the generating functions of the number of inversions between multisets, satisfy the beautiful identity

\[
\prod_{i=1}^{n} \left( \sum_{j=0}^{m} q^{j\cdot m(i-1)+\binom{i}{2}} \right) = \sum_{r=0}^{n \cdot m} \left[ \begin{bmatrix} n \cdot m \end{bmatrix} \right] q^{\binom{r}{2}} t^r.
\]

1. Inversions between multisets. Let \((m_1, \ldots, m_n)\) denote the multiset on \(\{1, \ldots, n\}\), in which the multiplicity of \(i\) is \(m_i\). The number of elements in \((m_1, \ldots, m_n)\) is \(m_1 + \cdots + m_n\) and is denoted by \(|(m_1, \ldots, m_n)|\). We abbreviate the multiset \((m_1, \ldots, m_n)\), where \(m_1 = \cdots = m_n = m\), to \((n \cdot m)\).

A multiset \((a_1, \ldots, a_n)\) of \((n \cdot m)\) satisfies \(a_i \leq m\), for \(i = 1, \ldots, n\), and it uniquely defines a complementary multiset \((\bar{a}_1, \ldots, \bar{a}_n)\) satisfying \(a_i + \bar{a}_i = m\) \((i = 1, \ldots, n)\).

An inversion between the multisets \((a_1, \ldots, a_n)\) and \((b_1, \ldots, b_n)\), in that order, is a pair \((i, j)\), where \(i\) is an element of \((a_1, \ldots, a_n)\), and \(j\) is an element of \((b_1, \ldots, b_n)\), and \(i > j\).

Let \((a_1, \ldots, a_n)\) be a multiset of \((n \cdot m)\). We denote by \(I(a_1, \ldots, a_n)\) the number of inversions between \((a_1, \ldots, a_n)\) and \((\bar{a}_1, \ldots, \bar{a}_n)\). We let \(\begin{bmatrix} n \cdot m \end{bmatrix} \) denote the generating function

\[
\sum_{|\{(a_1, \ldots, a_n)\}|=r} q^{I(a_1, \ldots, a_n)}.
\]

Our first theorem is a generalization of both the Gaussian recurrence for the q-binomial (Gaussian) coefficients [2, p. 100] and the polynomial coefficient recurrence [1, p. 77].

**Theorem 1** [3].

\[
\begin{bmatrix} n \cdot m \end{bmatrix} = \sum_{i=0}^{m} q^{i((n-1)m-(r-i))} \binom{n-1 \cdot m}{r-i}.
\]

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PROOF.

\[
\left[ \begin{array}{c} n \\ r \end{array} \right] = \sum_{\|a_1, \ldots, a_n\| = r} q^{I(a_1, \ldots, a_n)} = \sum_{i=0}^{m} \sum_{\|a_1, \ldots, a_{n-1}, i\| = r} q^{I(a_1, \ldots, a_{n-1}, i)}
\]

\[= \sum_{i=0}^{m} q_{i(a_1 + \cdots + a_{n-1})} \sum_{\|a_1, \ldots, a_{n-1}\| = r-i} q^{I(a_1, \ldots, a_{n-1})}
\]

\[= \sum_{i=0}^{m} q^{i((n-1)m-(r-i))} \left[ \begin{array}{c} (n-1) \cdot m \\ r-i \end{array} \right],
\]

where the third equality is explained by

\[I(a_1, a_2, \ldots, a_{n-1}, i)
\]

\[= a_2 a_1 + a_3(a_1 + a_2) + \cdots + a_{n-1}(a_1 + \cdots + a_{n-2}) + i(a_1 + \cdots + a_{n-1})
\]

\[= I(a_1, \ldots, a_{n-1}) + i(a_1 + \cdots + a_{n-1}). \quad \text{Q.E.D.}
\]

Because of Theorem 1, we will call \(\left[ \begin{array}{c} n \\ r \end{array} \right]\), a \(q\)-polynomial coefficient.

2. Identity.

THEOREM 2.

\[\prod_{i=1}^{n} \left( \sum_{j=0}^{m} q^{jm(i-1)+\binom{j}{2} t^j} \right) = \sum_{r=0}^{nm} \left[ \begin{array}{c} n \cdot m \\ r \end{array} \right] q^{\binom{r}{2} t^r}.
\]

PROOF. We prove the identity by induction on \(n\), using Theorem 1. The identity obviously holds when \(n = 1\). Assume that it holds for \(n - 1\). Then,

\[\prod_{i=0}^{n} \left( \sum_{j=0}^{m} q^{jm(i-1)+\binom{j}{2} t^j} \right)
\]

\[= \left( \prod_{i=0}^{n-1} \left( \sum_{j=0}^{m} q^{jm(i-1)+\binom{j}{2} t^j} \right) \right) \left( \sum_{j=0}^{m} q^{jm(n-1)+\binom{j}{2} t^j} \right)
\]

\[= \left( \sum_{k=0}^{m} \left[ \begin{array}{c} (n-1) \cdot m \\ k \end{array} \right] q^{\binom{k}{2} t^k} \right) \left( \sum_{j=0}^{m} q^{jm(n-1)+\binom{j}{2} t^j} \right)
\]

\[= \sum_{r=0}^{nm} \left( \sum_{i=0}^{m} \left[ \begin{array}{c} (n-1) \cdot m \\ r-i \end{array} \right] q^{i((n-1)m-(r-i))} \right) q^{\binom{r}{2} t^r}
\]

\[= \sum_{r=0}^{nm} \left[ \begin{array}{c} n \cdot m \\ r \end{array} \right] q^{\binom{r}{2} t^r}. \quad \text{Q.E.D.}
\]

If we set \(m = 1\) in Theorem 2, we obtain the well-known identity [2, p. 101]

\[(1 + t)(1 + tq) \cdots (1 + tq^{n-1}) = \sum_{r=0}^{n} \left[ \begin{array}{c} n \\ r \end{array} \right] q^{\binom{r}{2} t^r}.
\]
REFERENCES


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