PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 1, September 1985

SHORTER NOTES

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ON A PROBLEM OF BERMAN CONCERNING RADIAL LIMITS

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ABSTRACT. Given a G_δ-subset E of the unit circle T such that E is of measure zero, we prove that there exists a nonvanishing function $g \in H^\infty$ such that $g(z)$ has a radial limit at each point of T and this radial limit is zero at each point of E. This answers a problem of R. Berman (Proc. Amer. Math. Soc. **92** (1984), 64–66).

Let $U = \{z: |z| < 1\}$ and $T = \{z: |z| = 1\}$. Recently, R. Berman [**1**, Remark (1), p. 66] asked the following question: If E is a G_δ-subset of T such that E is of measure zero, does there exist a nonvanishing function $g \in H^\infty(U)$ such that $g(z)$ has a radial limit at each point of T and this radial limit is zero at each point of E? Previously, R. Cahill [**2**, Theorem 5, p. 171] proved that, under the same conditions on E, there exists a nonvanishing function $g \in H^\infty(U)$ such that the modulus of g has a radial limit at each point of T and this radial limit is zero at each point of E. In this note, we will give an affirmative answer to Berman's question by showing that Cahill's function g has a radial limit at each point of T.

THEOREM. *If E is a G_δ-subset of T such that E is of measure zero, then there exists a nonvanishing function $g \in H^\infty(U)$ such that $g(z)$ has a radial limit at each point of T and this radial limit is zero at each point of E.*

PROOF. Z. Zahorski [**5**] proved that there exists a monotone increasing differentiable function ϕ on $(-\pi, \pi)$ such that $\phi'(\theta) = Z(\theta)$ exists everywhere on $(-\pi, \pi)$ and $Z(\theta) = \infty$ if and only if $e^{i\theta} \in E$. In [**2**], Cahill extended $Z(\theta)$ to the whole real line so that $Z(\theta)$ is periodic with period 2π, and defined a function

$$h(z) = u(z) + iv(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} Z(\theta)\, d\theta$$

such that $u(z) > 0$ for each $z \in U$ and $u(z)$ has a radial limit at each point of T, where this radial limit is $+\infty$ at each point of E. For our purposes, the important point in Cahill's reasoning is that

$$(1) \qquad \lim_{h\to 0} \frac{1}{h}\int_0^h (Z(\theta + t) - Z(\theta - t))\, dt = 0 \quad \text{for each } \theta \in [-\pi, \pi].$$

Received by the editors December 6, 1984.
1980 *Mathematics Subject Classification*. Primary 30D40.
Key words and phrases. Radial limit.

(This was proved by Zahorski.) By a theorem due to Plessner [**3**], we have

$$v(re^{i\theta}) \to -\frac{1}{2\pi}\int_0^{\pi}(Z(\theta+t)-Z(\theta-t))\cot\frac{t}{2}\,dt \quad \text{as } r \to 1- \tag{2}$$

for each $\theta \in [-\pi, \pi]$ (see [**4**, Theorem IV.27, p. 162]). In fact, the argument in [**4**] shows that (1) implies that the right side of (2) is finite for each $\theta \in [-\pi, \pi]$, so $v(z)$ has a finite radial limit at each point of T. Cahill defined $g(z) = \exp(-h(z)) = \exp(-u(z) - iv(z))$, and thus $g(z)$ has a radial limit at each point of T, since $g(z)$ has the radial limit zero at each point of E, and both $u(z)$ and $v(z)$ have finite radial limits at each other point of T.

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