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SOME ALGEBRAICALLY INDEPENDENT CONTINUED FRACTIONS

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ABSTRACT. Using simple arguments, we prove algebraic independence of a class of continued fractions extending an earlier result of Bundschuh. We then apply it to give another proof of algebraic independence of numbers whose g-adic and continued fraction expansions are explicitly known.

1. Introduction.

Let

$$A = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+}\cdots = [a_0; a_1, a_2, a_3, \ldots]$$

and

$$B = b_0 + \frac{1}{b_1+}\frac{1}{b_2+}\frac{1}{b_3+}\cdots = [b_0; b_1, b_2, b_3, \ldots]$$

be continued fractions with positive integral partial quotients. Recently, Bundschuh [**3**] proved that if there exists a real number $r > 1$ such that

$$r^{-1}a_n \geq b_n \geq a_{n-1}^{n-1} \qquad (n = 1, 2, 3, \ldots),$$

then A and B are algebraically independent (over $\mathbf{Q}$). Bundschuh's proof makes use of a criterion for algebraic independence established by Durand [**4**]. In this note, we shall prove this same theorem under a weaker hypothesis. Our method does not appeal to Durand's criterion, but is based on an algebraic independence proof of Shiokawa [**7**] for gap series. Shiokawa mentioned that his arguments were essentially due to Flicker [**5**]. We now formulate our main result.

THEOREM. *Let $A, B, (a_n), (b_n)$ be as above. Let $r > 1$, (n_j) be an increasing sequence of positive integers and let $f(n)$ be an integer-valued function of natural argument n with $f(n) \geq 2$ $(n = 0, 1, 2, \ldots)$, and $f(n_j) \to \infty$ $(j \to \infty)$. If*

$$r^{-1}a_n \geq b_n \geq a_{n-1}^{f(n-1)} \qquad (n = 1, 2, 3, \ldots),$$

then A and B are algebraically independent (over $\mathbf{Q}$).

As an application, we shall prove

COROLLARY. *Let β be a positive irrational number, $g_1 \geq g_2$ be two distinct natural numbers > 1. If the simple continued fraction of β has unbounded partial quotients each of which is $\geq 1 + (2\log g_1)/\log g_2$, then the two numbers*

$$\sum_{i=1}^{\infty} (g_t - 1)g_t^{-[i\beta]} \qquad (t = 1, 2),$$

where $[x]$ denotes the integer part of x, are algebraically independent (over $\mathbf{Q}$).

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The two algebraically independent numbers in the corollary are interesting because their g-adic developments (as defined above) and their continued fractions can explicitly be computed. They have recently become better known owing to the works of Adams and Davison [**1**] and Bundschuh [**2**]. Their algebraic independence, under a slightly weaker hypothesis on the partial quotients, was first proved by Bundschuh [**2**] using Durand's criterion.

2. Lemma. We shall first recall an auxiliary result. For $A, B, (a_n), (b_n)$ as above, let the nth convergents of A, B be, respectively,

$$\frac{p_n(A)}{q_n(A)} = [a_0; a_1, \ldots, a_n], \quad \frac{p_n(B)}{q_n(B)} = [b_0; b_1, \ldots, b_n] \qquad (n = 0, 1, \ldots).$$

LEMMA. *Let the notation be as above. Then*

(i)

$$\begin{aligned} &0 < |A - p_n(A)/q_n(A)| \le (q_n(A)q_{n+1}(A))^{-1} \le (a_{n+1}q_n^2(A))^{-1}, \\ &0 < (2q_n(B)q_{n+1}(B))^{-1} \le |B - p_n(B)/q_n(B)| \\ &\quad \le (q_n(B)q_{n+1}(B))^{-1} \le (b_{n+1}q_n^2(B))^{-1}. \end{aligned}$$

(ii) *If* $a_n \ge rb_n$ $(r > 1)$ *for all* $n = 1, 2, \ldots,$ *then*

$$q_n(A) \ge r^{n/2} q_n(B) \quad (n = 0, 1, 2, \ldots).$$

(iii) *If* (a_n) *is a strictly increasing sequence and* $a_{n+1} \ge a_n^2$ $(n = 2, 3, 4, \cdots)$, *then*

$$q_n(A) \le a_n^2 \quad (n = 0, 1, 2, \ldots).$$

PROOF. (i) is standard for any continued fractions; see e.g. Chapter X of Hardy and Wright [**6**].

(ii) is Lemma 3 of Bundschuh [**3**]; it follows easily from induction.

(iii) is Lemma 1(iv) of Bundschuh [**3**]; again the proof is easily done by induction.

3. Proof of the Theorem. Suppose on the contrary that A and B are algebraically dependent (over $\mathbf{Q}$). Then there exits a nonidentically vanishing

$$P(X, Y) := \sum_{i=0}^{D_1} \sum_{j=0}^{D_2} w_{ij} X^i Y^j \in \mathbf{Z}[X, Y]$$

such that $P(A, B) = 0$. We may assume that $P(X, Y)$ is one with minimum total degree $D_1 + D_2$ among such polynomials. Consider for fixed natural number n

$$P_n := P\left(\frac{p_n(A)}{q_n(A)}, \frac{p_n(B)}{q_n(B)}\right) = \sum_{i,j} w_{ij} \left(\frac{p_n(A)}{q_n(A)}\right)^i \left(\frac{p_n(B)}{q_n(B)}\right)^j.$$

If $P_n \ne 0$, then being a rational number we must have

$$|P_n| \ge (q_n(A))^{-D_1}(q_n(B))^{-D_2}$$

and so, using part (ii) of the lemma,

$$|P_n| \ge r^{nD_2/2}(q_n(A))^{-D_1-D_2}. \tag{1}$$

Let

$$\delta_1 := \delta_1(n, A) = A - p_n(A)/q_n(A),$$
$$\delta_2 := \delta_2(n, B) = B - p_n(B)/q_n(B).$$

Then

$$\begin{aligned} P_n &= P(A-\delta_1, B-\delta_2) = \sum_{i,j} w_{ij}(A-\delta_1)^i(B-\delta_2)^j \\ &= w_1\delta_1 + w_2\delta_2 + O(|\delta|^2) \end{aligned} \tag{2}$$

where $|\delta| = \max(|\delta_1|, |\delta_2|)$,

$$w_1 = -\sum_{i,j} iw_{ij}A^{i-1}B^j, \quad \text{and} \quad w_2 = -\sum_{i,j} jw_{ij}A^iB^{j-1}.$$

By part (i) of the lemma and by the hypotheses of the theorem, we get

$$|\delta_1| = |A - p_n(A)/q_n(A)| \le (a_{n+1}q_n^2(A))^{-1} \le (ra_n^{f(n)}q_n^2(A))^{-1},$$

and so by part (iii) of the lemma,

$$|\delta_1| \le r^{-1}(q_n(A))^{-2-f(n)/2}. \tag{3}$$

Similarly,

$$\begin{aligned} |\delta_2| &= |B - p_n(B)/q_n(B)| \le (b_{n+1}q_n^2(B))^{-1} \\ &\le (a_n^{f(n)}q_n^2(B))^{-1} \le (q_n(A))^{-f(n)/2}(q_n^2(B))^{-1}. \end{aligned} \tag{4}$$

Thus from (1)–(4), if $P_n \neq 0$, we have

$$r^{nD_2/2}(q_n(A))^{-D_1-D_2} \le O((q_n(A))^{-f(n)/2})$$

which is a contradiction when $n \in (n_j)$ is sufficiently large. To complete the proof, we need only show that $P_n \neq 0$ for infinitely many $n \in (n_j)$. First we observe that at least one of the w_1, w_2 is $\neq 0$. For if not, unless $P(X, Y)$ is constant with respect to one of its arguments, A and B would satisfy a polynomial equation of total degree lower than $D_1 + D_2$, namely,

$$\sum_{i,j} iw_{ij}X^{i-1}Y^j = 0 \quad \text{or} \quad \sum_{i,j} jw_{ij}X^iY^{j-1} = 0.$$

Since it involves only minor change of arguments for the case where only one of the w_1, w_2 is 0, we shall assume that none of the w_1, w_2 is 0. We also observe that from parts (i) and (ii) of the lemma,

$$\left|\frac{\delta_1}{\delta_2}\right| \le \frac{2q_n(B)q_{n+1}(B)}{q_n(A)q_{n+1}(A)} \le 2(r^{n/2+(n+1)/2})^{-1} \to 0 \quad (n \in (n_j),\ n \to \infty).$$

Consequently, $P_n = \delta_2 Q$ where

$$Q = w_1\delta_1/\delta_2 + w_2 + O(|\delta|^2/\delta_2) \to w_2 \neq 0 \quad (n \in (n_j),\ n \to \infty)$$

and so $P_n \neq 0$ for infinitely many $n \in (n_j)$. This completes the proof of the theorem.

4. Proof of the Corollary. Let

$$A = \sum_{i=1}^{\infty} (g_1 - 1) g_1^{-[i\beta]}, \quad B = \sum_{i=1}^{\infty} (g_2 - 1) g_2^{-[i\beta]},$$

and let the continued fractions of A, B, β^{-1} be

$$[a_0; a_1, a_2, \ldots], \; [b_0; b_1, b_2, \ldots], \; [c_0; c_1, c_2, \ldots],$$

respectively. From Bundschuh [**2**, Theorem 1, p. 111], we know

$$\left. \begin{aligned} a_n &= g_1^{q_{n-2}} \sum_{i=0}^{c_n - 1} g_1^{i q_{n-1}}, \\ b_n &= g_2^{q_{n-2}} \sum_{i=0}^{c_n - 1} g_2^{i q_{n-2}}, \end{aligned} \right\} \quad (n = 1, 2, 3, \ldots)$$

where q_n denotes the denominator of the nth convergent of the continued fraction of β^{-1}. Since $g_1 > g_2$, then

$$\begin{aligned} a_n &\geq g_1^{q_{n-2}} \sum_{i=0}^{c_n - 1} g_2^{i q_{n-2}} \\ &= (g_1/g_2)^{q_{n-2}} g_2^{q_{n-2}} \sum_{i=0}^{c_n - 1} g_2^{i q_{n-2}} \quad (n = 1, 2, 3, \ldots) \\ &\geq (g_1/g_2) g_2^{q_{n-2}} \sum_{i=0}^{c_n - 1} g_2^{i q_{n-2}} = (g_1/g_2) b_n, \end{aligned}$$

and so there exists $r = g_1/g_2 > 1$ such that $r^{-1} a_n \geq b_n$ $(n = 1, 2, 3, \ldots)$. Also, from equation (10), p. 114 of Bundschuh [**2**], we know that

$$\begin{aligned} b_n &\geq g_2^{(c_n - 1) q_{n-1}} \qquad (n = 1, 2, 3, \ldots), \\ a_{n-1} &< g_1^{c_{n-1} q_{n-2}} \qquad (n = 4, 5, 6, \ldots). \end{aligned}$$

In order to complete the proof, it suffices to show that for $n \geq 4$ there exists an integer-valued function $f(n)$ with $f(n) \geq 2$, $f(n_j) \to \infty$ for some increasing subsequence (n_j) of positive integers, such that $b_n \geq a_{n-1}^{f(n-1)}$. To satisfy this last inequality, it is enough from the above estimates to have

$$f(n-1) \leq \frac{(c_n - 1) q_{n-1} \log g_2}{c_{n-1} q_{n-2} \log g_1} = \frac{\log g_2}{\log g_1} (c_n - 1) \left(1 + \frac{q_{n-3}}{c_{n-1} q_{n-2}} \right).$$

Choose

$$f(n-1) = \left[\frac{\log g_2}{\log g_1} (c_n - 1) \right] \quad (n = 4, 5, 6, \ldots).$$

From the hypotheses that (c_n) is an unbounded sequence each of which is $\geq 1 + 2 \log g_1 / \log g_2$, we immediately infer that $f(n) \geq 2$ and $f(n_j) \to \infty$ for some increasing sequence of positive integers (n_j).

REFERENCES

1. W. W. Adams and J. L. Davison, *A remarkable class of continued fractions*, Proc. Amer. Math. Soc. **65** (1977), 194–198.
2. P. Bundschuh, *Über eine Klasse reeller transzendenter Zahlen mit explizit angebbarer g-adischer und Kettenbruch-Entwicklung*, J. Reine Angew. Math. **318** (1980), 110–119.
3. ——, *Transcendental continued fractions*, J. Number Theory **18** (1984), 91–98.
4. A. Durand, *Indépendance algébrique de nombres complexes et critére de transcendance*, Compositio Math. **35** (1977), 259–267.
5. Y. Flicker, *Algebraic independence by a method of Mahler*, J. Austral. Math. Soc. Ser. A **27** (1979), 173–188.
6. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers* (4th ed.), Oxford Univ. Press, London, 1971.
7. I. Shiokawa, *Algebraic independence of certain gap series*, Arch. Math. **38** (1982), 438–442.

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