

## SINGULAR SOLUTIONS OF THE HEAT EQUATION WITH ABSORPTION

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ABSTRACT. In this paper we prove that the source-type solutions converge—when the total initial mass tends to infinity—to the very singular solution obtained in [3].

**1. Introduction.** In a recent paper [3], Brezis, Peletier and Terman found a *very singular solution* (VSS) of the equation

$$(1) \quad u_t = \Delta u - u^p \quad \text{in } S = \mathbf{R}^n \times (0, \infty)$$

when  $p < (n + 2)/n$ . By this they meant a function  $W(x, t)$  with the properties

- (i)  $W$  is smooth in  $\bar{S}$ , except at  $(0, 0)$ ;
- (ii)  $W > 0$  in  $S$ ;
- (iii)  $W(x, 0) = 0$  for all  $x \in \mathbf{R}^n \setminus \{0\}$ ;
- (iv)  $W$  is more singular than the fundamental solution  $E$  of the heat equation

$$(2) \quad E(x, t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t);$$

specifically

$$\int_{\mathbf{R}^n} W(x, t) dx \rightarrow \infty \quad \text{as } t \downarrow 0.$$

The function  $W$  they found was of the form

$$(3) \quad W(x, t) = t^{-1/(p-1)} f(|x|/t^{1/2})$$

where  $f(\eta)$  is a solution of the problem

$$(P) \quad \begin{cases} f'' + \left( \frac{n-1}{\eta} + \frac{\eta}{2} \right) f' + \frac{1}{p-1} f - f^p = 0 & \text{on } (0, \infty), \\ f > 0 \text{ and } f \text{ is smooth} & \text{on } [0, \infty), \\ f'(0) = 0 \text{ and } \lim_{\eta \rightarrow \infty} \eta^{2/(p-1)} f(\eta) = 0. \end{cases}$$

It was shown that this solution  $f$  is unique, and that it behaves as  $\eta \rightarrow \infty$  like

$$(4) \quad f(\eta) = c_0 \exp(-\eta^2/4) \eta^{2/(p-1)-n} [1 + O(\eta^{-2})],$$

where  $c_0$  is a known positive constant.

Besides this very singular solution, equation (1) has a one parameter family of solutions  $V_c$ ,  $c > 0$ , which we shall call *singular solutions* (SS). They share the properties (i)–(iii) with  $W$ , but

$$(iv^*) \quad V_c(x, 0) = c\delta(x) \text{ in } \mathbf{R}^n.$$

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It was shown by Brezis and Friedman [1] that when  $p < (n + 2)/n$ , the solution  $V_c$  exists and is unique for any  $c > 0$ . In addition, it was shown in [3] that

$$|V_c(x, t) - cE(x, t)| \leq Ac^p t^\nu E(x, t) \quad \text{for } (x, t) \in S$$

where  $A > 0$  and  $\nu = (n/2)((n + 2)/n - p)$ , and hence that  $V_c$  has the same singularity at  $(0, 0)$  as  $E$ .

It is the object of this paper to show that, if  $p < (n + 2)/n$ , the very singular solution  $W$  can be obtained as the limit of singular solutions  $V_c$  as  $c \rightarrow \infty$ .

**THEOREM 1.** *Let  $p < (n + 2)/n$ . Then*

$$\lim_{c \rightarrow \infty} V_c(x, t) = W(x, t) \quad \text{for } (x, t) \in \bar{S} \setminus \{(0, 0)\}$$

*uniformly on compact sets.*

An analogous situation exists for the elliptic equation

$$(5) \quad -\Delta u + u^p = c\delta(x) \quad \text{in } \mathbf{R}^n$$

in which  $c > 0$ . For  $p < n/(n - 2)$ , equation (5) has a uniquely determined very singular solution  $W$ , which can be obtained as the limit of singular solutions  $V_c$ , which behave near the origin like  $cE$ , where  $E$  is now the fundamental solution of  $-\Delta$  [2, 4, 7, 8].

As a by-product of the proof of Theorem 1 we shall obtain the following asymptotic result:

**THEOREM 2.** *Let  $p < (n + 2)/n$ . Then for each  $c > 0$ ,*

$$\lim_{t \rightarrow \infty} t^{1/(p-1)} V_c(x, t) = f(|x|/t^{1/2}),$$

*where  $f$  is the solution of Problem (P), uniformly on the family of sets  $P_a$  defined by*

$$P_a = \{x \in \mathbf{R}^n : |x| \leq at^{1/2}\}, \quad a > 0.$$

**2. SS  $\rightarrow$  VSS.** Consider the problem

$$(I) \quad \begin{cases} u_t = \Delta u - u^p & \text{in } S, \\ u(x, 0) = h(x) & \text{in } \mathbf{R}^n, \end{cases}$$

in which  $h$  is a nonnegative function in  $L^\infty(\mathbf{R}^n)$ . It is well known [6] that this problem has a unique solution, which satisfies (1) in a classical sense.

We begin with a result taken from Brezis and Friedman [1, p. 83].

**PROPOSITION.** *Suppose  $p < (n + 2)/n$ . Let  $\{h_l\}$  be a sequence in  $L^\infty(\mathbf{R}^n)$  such that  $h_l > 0$  and*

$$h_l(x) \rightarrow c\delta(x) \quad \text{as } l \rightarrow \infty \text{ in } \mathcal{D}',$$

*with  $c > 0$ , and let  $\{v_l\}$  be the sequence of solutions of Problem (I) corresponding to  $h_l$ . Then*

$$v_l \rightarrow V_c \quad \text{as } l \rightarrow \infty$$

*uniformly on compact subsets of  $\bar{S} \setminus \{(0, 0)\}$ .*

We use this Proposition to derive two properties of the family of solutions  $\{V_c : c > 0\}$ .

LEMMA 1. Let  $0 < c_1 < c_2 < \infty$ . Then

$$V_{c_1}(x, t) \leq V_{c_2}(x, t) \quad \text{for } (x, t) \in \bar{S} \setminus \{(0, 0)\}.$$

PROOF. Let  $h_l(x) = E(x, 1/l)$ , where  $E$  is defined in (2) and  $l > 0$ . Then, by the properties of  $E$ ,

$$h_l(x) \rightarrow \delta(x) \quad \text{as } l \rightarrow \infty \text{ in } \mathcal{D}'.$$

Let  $v_{1l}$  and  $v_{2l}$  be the solutions of Problem (I) which correspond to, respectively,  $c_1 h_l$  and  $c_2 h_l$ . Then by the maximum principle, for all  $l > 0$ ,

$$(6) \quad v_{1l}(x, t) \leq v_{2l}(x, t) \quad \text{in } \bar{S}.$$

By the Proposition,

$$v_{il}(x, t) \rightarrow V_{c_i}(x, t) \quad \text{as } l \rightarrow \infty \quad (i = 1, 2)$$

for every  $(x, t) \in \bar{S} \setminus \{(0, 0)\}$ , whence, in view of (6),

$$V_{c_1}(x, t) \leq V_{c_2}(x, t) \quad \text{for } (x, t) \in \bar{S} \setminus \{(0, 0)\}.$$

LEMMA 2. For any  $c > 0$ ,

$$V_c(x, t) \leq W(x, t) \quad \text{for } \bar{S} \setminus \{(0, 0)\}.$$

PROOF. Observe that

$$(7) \quad \int_{\mathbf{R}^n} W(x, t) dx = |S_1| t^{-1/(p-1)+n/2} \int_0^\infty f(\eta) \eta^{n-1} d\eta,$$

where  $|S_1|$  denotes the surface area of the unit ball. Note that the integral on the right of (7) converges in view of (4).

Remembering that  $p < (n+2)/n$ , we conclude that, for every  $c > 0$ , there exists a unique  $\tau_c > 0$  such that

$$\int_{\mathbf{R}^n} W(x, \tau_c) dx = c.$$

For  $A > 0$  we define the truncated VSS

$$W_A(x, t) = \begin{cases} W(x, t) & \text{if } W(x, t) < A, \\ A & \text{if } W(x, t) \geq A. \end{cases}$$

Then by (7) there exists for any  $\tau \in (0, \tau_c)$  a unique positive number  $A(\tau)$  such that  $\int_{\mathbf{R}^n} W_{A(\tau)}(x, \tau) dx = c$ .

Let us now define

$$h_l(x) = W_{A(1/l)}(x, 1/l), \quad l = N, N+1, \dots,$$

where  $N$  has been chosen so that  $1/N < \tau_c$ , and let—as before— $v_l$  be the solution of Problem (I) which corresponds to the initial value  $h_l$ . Then, by construction,  $h_l(x) \leq W(x, 1/l)$  for  $x \in \mathbf{R}^n$ , and hence, by the maximum principle,

$$(8) \quad v_l(x, t) \leq W(x, 1/l) \quad \text{for } (x, t) \in \bar{S}.$$

Now we let  $l \rightarrow \infty$ . Then by the construction of the sequence  $\{h_l\}$  and the properties of  $W$ ,  $h_l(x) \rightarrow c\delta(x)$  as  $l \rightarrow \infty$  in  $\mathcal{D}'$ , whence, by the Proposition,

$$(9) \quad v_l(x, t) \rightarrow V_c(x, t) \quad \text{for } (x, t) \in \bar{S} \setminus \{(0, 0)\}.$$

On the other hand, it is readily seen from the expression (3) for  $W$  that

$$(10) \quad W(x, t + 1/l) \rightarrow W(x, t) \quad \text{as } l \rightarrow \infty, \text{ for } (x, t) \in \overline{S} \setminus \{(0, 0)\}.$$

Thus, putting (8), (9) and (10) together we find that

$$V_c(x, t) \leq W(x, t) \quad \text{for } (x, t) \in \overline{S} \setminus \{(0, 0)\}.$$

We deduce from Lemmas 1 and 2 the following corollary:

COROLLARY 3. *For every  $(x, t) \in \overline{S} \setminus \{(0, 0)\}$ ,  $\lim_{c \rightarrow \infty} V_c(x, t)$  exists.*

Thus, we can define the function

$$(11) \quad U(x, t) \stackrel{\text{def}}{=} \lim_{c \rightarrow \infty} V_c(x, t), \quad (x, t) \in \overline{S} \setminus \{(0, 0)\}.$$

In the remainder of this section we shall prove that  $U = W$ .

LEMMA 4. *The function  $U$  defined in (11) has the properties*

- (i)  $U \in C^{2,1}(S) \cap C(\overline{S} \setminus \{(0, 0)\})$ ;
- (ii)  $U_t = \Delta U - U^p$  in  $S$ ;
- (iii)  $U(x, t) > 0$  for  $(x, t) \in S$ ;
- (iv)  $U(x, 0) = 0$  for  $x \in \mathbf{R}^n \setminus \{(0, 0)\}$ ;
- (v)  $\int_{\mathbf{R}^n} U(x, t) dx \rightarrow \infty$  as  $t \downarrow 0$ ;
- (vi)  $U(x, t) = \hat{U}(|x|, t)$ .

PROOF. By Lemma 2, the set of solutions  $\{V_c: c > 0\}$  of equation (1) is locally bounded in  $S$ . Hence, by standard (interior) regularity theory [6], their limit  $U$  belongs to  $C^{2,1}(S)$  and satisfies equation (1). Because the functions  $V_c$  are all positive and increasing in  $c$ ,  $U > 0$  in  $S$ . Next, since for every  $c > 0$ ,

$$(12) \quad 0 \leq V_c \leq W \quad \text{in } \overline{S} \setminus \{(0, 0)\}$$

and  $W(x, 0) = 0$  for  $x \in \mathbf{R}^n \setminus \{0\}$ ,  $W$  is a barrier function at  $t = 0$ , ensuring that  $U \in C(\overline{S} \setminus \{(0, 0)\})$  and that  $U(x, 0) = 0$  for all  $x \in \mathbf{R}^n \setminus \{0\}$ .

To prove property (v), we observe that, for any  $c > 0$ ,  $\int U(x, t) dx \geq \int V_c(x, t) dx$  and hence

$$\liminf_{t \downarrow 0} \int U(x, t) dx \geq \lim_{t \downarrow 0} \int V_c(x, t) dx = c,$$

the integrals being taken over  $\mathbf{R}^n$ . Since  $c$  may be chosen arbitrarily large, the result follows. Finally, the symmetry property (vi) follows from the fact that the functions  $V_c$  are all endowed with this property.

We know that the function  $W$  has all the properties listed for  $U$  in Lemma 4, and we expect it is the *only* function with these properties. However, as yet we have not seen a proof.

In the present case, we can deduce from certain scaling properties of the functions  $V_c$  that  $U$  must be a similarity solution. This is the content of the next lemma.

LEMMA 5. *The function  $U$  can be expressed in the form*

$$U(x, t) = t^{-1/(p-1)} g(\eta), \quad \eta = |x|/t^{1/2}.$$

PROOF. Set  $u(x, t) = V_1(x, t)$ , and define the family of functions

$$(13) \quad u_k(x, t) = k^{2/(p-1)} u(kx, k^2t), \quad k > 0.$$

Then for each  $k > 0$ ,  $u_k$  is a solution of the problem

$$(14) \quad \begin{aligned} u_t &= \Delta u - u^p && \text{in } S, \\ u(x, 0) &= k^{2/(p-1)}\delta(kx) && \text{in } \mathbf{R}^n. \end{aligned}$$

Observe that the initial condition (14) may be replaced by

$$u(x, 0) = k^{2/(p-1)-n}\delta(x).$$

Thus, defining the function  $c(k) = k^{2/(p-1)-n}$ , we obtain, in view of the uniqueness of  $V_c$ ,

$$u_k(x, t) = V_{c(k)}(x, t).$$

Note that since  $p < (n+2)/n$ ,  $c(k) \rightarrow \infty$  as  $k \rightarrow \infty$ , and hence,

$$(15) \quad u_k(x, t) \rightarrow U(x, t) \quad \text{as } k \rightarrow \infty \text{ for } (x, t) \in \bar{S} \setminus \{(0, 0)\}.$$

Now let  $l > 0$ . Then

$$(16) \quad u_{kl}(x, t) = (kl)^{2/(p-1)}u(klx, (kl)^2t) = l^{2/(p-1)}u_k(lx, l^2t).$$

Hence, if we let  $k \rightarrow \infty$  in (16) and we use (15) we obtain

$$(17) \quad U(x, t) = l^{1/(p-1)}U(lx, l^2t).$$

Finally, setting  $l = t^{-1/2}$ , we obtain

$$U(x, t) = t^{-1/(p-1)}U(x/t^{1/2}, 1) = t^{-1/(p-1)}\hat{U}(|x|/t^{1/2}, 1)$$

which is the desired form if we set  $g(r) = \hat{U}(r, 1)$ .

It is easily verified that the properties (i)–(vi) of  $U$ , listed in Lemma 4, ensure that  $g$  is a solution of Problem (P). By the uniqueness theorem for this problem [3] we may conclude that  $g = f$  and hence that  $U = W$ . This proves Theorem 1.

REMARK. Note that the proof of Lemma 5 hinges on two invariance properties of  $U$ :

- (a) invariance with respect to scaling: see (17);
- (b) invariance with respect to rotations: see Lemma 4(vi).

The fact that we have a uniqueness theorem for precisely this class of functions enables us to state that  $U = W$ .

Using the scaling method of [5], we can deduce from (13) the limiting behaviour of  $V_c(x, t)$  as  $t \rightarrow \infty$ . For if we set  $t = 1$  and let  $k \rightarrow \infty$  in (13) we obtain

$$\lim_{k \rightarrow \infty} k^{2/(p-1)}V_c(kx, k^2) = U(x, 1)$$

uniformly on bounded sets in  $\mathbf{R}^n$ . Thus, with  $k^2 = s$  and  $kx = y$ , we find that

$$\lim_{s \rightarrow \infty} s^{1/(p-1)}V_c(y, s) = U(y/s^{1/2}, 1) = f(|y|/s^{1/2}),$$

uniformly on sets  $P_a = \{(y, s): |y| \leq as^{1/2}\}$ ,  $a > 0$ . This proves Theorem 2.

## REFERENCES

1. H. Brezis and A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions*, J. Math. Pures Appl. **62** (1983), 73–97.
2. H. Brezis and E. Lieb, *Long range potentials in Thomas-Fermi theory*, Comm. Math. Phys. **65** (1979), 231–246.
3. H. Brezis, L. A. Peletier and D. Terman, *A very singular solution of the heat equation with absorption*, Arch. Rational Mech. Anal. (to appear).
4. H. Brezis and L. Veron, *Removable singularities of some nonlinear elliptic equations*, Arch. Rational Mech. Anal. **75** (1980), 1–6.
5. S. Kamin and L. A. Peletier, *Large time behavior of solutions of the heat equation with absorption*, Ann. Scuola Norm. Sup. Pisa (to appear).
6. O. A. Oleinik and S. N. Kruzhkov, *Quasilinear second order parabolic equations with many independent variables*, Russian Math. Surveys **16** (1961), 105–146.
7. L. Veron, *Singular solutions of some nonlinear equations*, Nonlinear Anal. **5** (1981), 225–242.
8. ———, *Weak and strong singularities of nonlinear elliptic equations*, Proc. Sympos. Pure Math. (to appear).

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