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SINGULAR SOLUTIONS OF THE HEAT EQUATION WITH ABSORPTION

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ABSTRACT. In this paper we prove that the source-type solutions converge—when the total initial mass tends to infinity—to the very singular solution obtained in [**3**].

1. Introduction. In a recent paper [**3**], Brezis, Peletier and Terman found a *very singular solution* (VSS) of the equation

$$u_t = \Delta u - u^p \quad \text{in } S = \mathbf{R}^n \times (0,\infty) \tag{1}$$

when $p < (n+2)/n$. By this they meant a function $W(x,t)$ with the properties

(i) W is smooth in $\overline{S}$, except at $(0,0)$;

(ii) $W > 0$ in S;

(iii) $W(x,0) = 0$ for all $x \in \mathbf{R}^n \backslash \{0\}$;

(iv) W is more singular than the fundamental solution E of the heat equation

$$E(x,t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t); \tag{2}$$

specifically

$$\int_{\mathbf{R}^n} W(x,t)\,dx \to \infty \quad \text{as } t \downarrow 0.$$

The function W they found was of the form

$$W(x,t) = t^{-1/(p-1)} f(|x|/t^{1/2}) \tag{3}$$

where $f(\eta)$ is a solution of the problem

$$\text{(P)} \qquad \begin{cases} f'' + \left(\dfrac{n-1}{\eta} + \dfrac{\eta}{2}\right) f' + \dfrac{1}{p-1} f - f^p = 0 & \text{on } (0,\infty), \\ f > 0 \text{ and } f \text{ is smooth} & \text{on } [0,\infty), \\ f'(0) = 0 \text{ and } \lim_{\eta\to\infty} \eta^{2/(p-1)} f(\eta) = 0. \end{cases}$$

It was shown that this solution f is unique, and that it behaves as $\eta \to \infty$ like

$$f(\eta) = c_0 \exp(-\eta^2/4)\eta^{2/(p-1)-n}[1 + O(\eta^{-2})], \tag{4}$$

where c_0 is a known positive constant.

Besides this very singular solution, equation (1) has a one parameter family of solutions V_c, $c > 0$, which we shall call *singular solutions* (SS). They share the properties (i)–(iii) with W, but

(iv*) $V_c(x,0) = c\delta(x)$ in $\mathbf{R}^n$.

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It was shown by Brezis and Friedman [**1**] that when $p < (n+2)/n$, the solution V_c exists and is unique for any $c > 0$. In addition, it was shown in [**3**] that

$$|V_c(x,t) - cE(x,t)| \le Ac^p t^\nu E(x,t) \quad \text{for } (x,t) \in S$$

where $A > 0$ and $\nu = (n/2)((n+2)/n - p)$, and hence that V_c has the same singularity at $(0,0)$ as E.

It is the object of this paper to show that, if $p < (n+2)/n$, the very singular solution W can be obtained as the limit of singular solutions V_c as $c \to \infty$.

THEOREM 1. *Let $p < (n+2)/n$. Then*

$$\lim_{c\to\infty} V_c(x,t) = W(x,t) \quad \textit{for } (x,t) \in \overline{S}\setminus\{(0,0)\}$$

uniformly on compact sets.

An analogous situation exists for the elliptic equation

$$-\Delta u + u^p = c\delta(x) \quad \text{in } \mathbf{R}^n \tag{5}$$

in which $c > 0$. For $p < n/(n-2)$, equation (5) has a uniquely determined very singular solution W, which can be obtained as the limit of singular solutions V_c, which behave near the origin like cE, where E is now the fundamental solution of $-\Delta$ [**2**, **4**, **7**, **8**].

As a by-product of the proof of Theorem 1 we shall obtain the following asymptotic result:

THEOREM 2. *Let $p < (n+2)/n$. Then for each $c > 0$,*

$$\lim_{t\to\infty} t^{1/(p-1)} V_c(x,t) = f(|x|/t^{1/2}),$$

where f is the solution of Problem (P), *uniformly on the family of sets P_a defined by*

$$P_a = \{x \in \mathbf{R}^n : |x| \le at^{1/2}\}, \qquad a > 0.$$

2. SS $\to$ VSS. Consider the problem

$$\text{(I)} \qquad \begin{cases} u_t = \Delta u - u^p & \text{in } S, \\ u(x,0) = h(x) & \text{in } \mathbf{R}^n, \end{cases}$$

in which h is a nonnegative function in $L^\infty(\mathbf{R}^n)$. It is well known [**6**] that this problem has a unique solution, which satisfies (1) in a classical sense.

We begin with a result taken from Brezis and Friedman [**1**, p. 83].

PROPOSITION. *Suppose $p < (n+2)/n$. Let $\{h_l\}$ be a sequence in $L^\infty(\mathbf{R}^n)$ such that $h_l > 0$ and*

$$h_l(x) \to c\delta(x) \quad \textit{as } l \to \infty \textit{ in } \mathcal{D}',$$

with $c > 0$, and let $\{v_l\}$ be the sequence of solutions of Problem (I) *corresponding to h_l. Then*

$$v_l \to V_c \quad \textit{as } l \to \infty$$

uniformly on compact subsets of $\overline{S}\setminus\{(0,0)\}$.

We use this Proposition to derive two properties of the family of solutions $\{V_c : c > 0\}$.

LEMMA 1. *Let $0 < c_1 < c_2 < \infty$. Then*

$$V_{c_1}(x,t) \le V_{c_2}(x,t) \quad \textit{for } (x,t) \in \overline{S}\backslash\{(0,0)\}.$$

PROOF. Let $h_l(x) = E(x, 1/l)$, where E is defined in (2) and $l > 0$. Then, by the properties of E,

$$h_l(x) \to \delta(x) \quad \text{as } l \to \infty \text{ in } \mathcal{D}'.$$

Let v_{1l} and v_{2l} be the solutions of Problem (I) which correspond to, respectively, $c_1 h_l$ and $c_2 h_l$. Then by the maximum principle, for all $l > 0$,

$$v_{1l}(x,t) \le v_{2l}(x,t) \quad \text{in } \overline{S}. \tag{6}$$

By the Proposition,

$$v_{il}(x,t) \to V_{c_i}(x,t) \quad \text{as } l \to \infty \ (i = 1, 2)$$

for every $(x,t) \in \overline{S}\backslash\{(0,0)\}$, whence, in view of (6),

$$V_{c_1}(x,t) \le V_{c_2}(x,t) \quad \text{for } (x,t) \in \overline{S}\backslash\{(0,0)\}.$$

LEMMA 2. *For any $c > 0$,*

$$V_c(x,t) \le W(x,t) \quad \textit{for } \overline{S}\{(0,0)\}.$$

PROOF. Observe that

$$\int_{\mathbf{R}^n} W(x,t)\,dx = |S_1| t^{-1/(p-1)+n/2} \int_0^\infty f(\eta)\eta^{n-1}\,d\eta, \tag{7}$$

where $|S_1|$ denotes the surface area of the unit ball. Note that the integral on the right of (7) converges in view of (4).

Remembering that $p < (n+2)/n$, we conclude that, for every $c > 0$, there exists a unique $\tau_c > 0$ such that

$$\int_{\mathbf{R}^n} W(x, \tau_c)\,dx = c.$$

For $A > 0$ we define the truncated VSS

$$W_A(x,t) = \begin{cases} W(x,t) & \text{if } W(x,t) < A, \\ A & \text{if } W(x,t) \ge A. \end{cases}$$

Then by (7) there exists for any $\tau \in (0, \tau_c)$ a unique positive number $A(\tau)$ such that $\int_{\mathbf{R}^n} W_{A(\tau)}(x,\tau)\,dx = c$.

Let us now define

$$h_l(x) = W_{A(1/l)}(x, 1/l), \quad l = N, N+1, \dots,$$

where N has been chosen so that $1/N < \tau_c$, and let—as before—v_l be the solution of Problem (I) which corresponds to the initial value h_l. Then, by construction, $h_l(x) \le W(x, 1/l)$ for $x \in \mathbf{R}^n$, and hence, by the maximum principle,

$$v_l(x,t) \le W(x, 1/l) \quad \text{for } (x,t) \in \overline{S}. \tag{8}$$

Now we let $l \to \infty$. Then by the construction of the sequence $\{h_l\}$ and the properties of W, $h_l(x) \to c\delta(x)$ as $l \to \infty$ in $\mathcal{D}'$, whence, by the Proposition,

$$v_l(x,t) \to V_c(x,t) \quad \text{for } (x,t) \in \overline{S}\backslash\{(0,0)\}. \tag{9}$$

On the other hand, it is readily seen from the expression (3) for W that

(10) $$W(x,t+1/l) \to W(x,t) \quad \text{as } l \to \infty, \text{ for } (x,t) \in \overline{S} \setminus \{(0,0)\}.$$

Thus, putting (8), (9) and (10) together we find that

$$V_c(x,t) \le W(x,t) \quad \text{for } (x,t) \in \overline{S} \setminus \{(0,0)\}.$$

We deduce from Lemmas 1 and 2 the following corollary:

COROLLARY 3. *For every* $(x,t) \in \overline{S} \setminus \{(0,0)\}$, $\lim_{c\to\infty} V_c(x,t)$ *exists.*

Thus, we can define the function

(11) $$U(x,t) \overset{\text{def}}{=} \lim_{c\to\infty} V_c(x,t), \qquad (x,t) \in \overline{S} \setminus \{(0,0)\}.$$

In the remainder of this section we shall prove that $U = W$.

LEMMA 4. *The function U defined in* (11) *has the properties*
(i) $U \in C^{2,1}(S) \cap C(\overline{S} \setminus \{(0,0)\})$;
(ii) $U_t = \Delta U - U^p$ *in* S;
(iii) $U(x,t) > 0$ *for* $(x,t) \in S$;
(iv) $U(x,0) = 0$ *for* $x \in \mathbf{R}^n \setminus \{(0,0)\}$;
(v) $\int_{\mathbf{R}^n} U(x,t)\,dx \to \infty$ *as* $t \downarrow 0$;
(vi) $U(x,t) = \hat{U}(|x|,t)$.

PROOF. By Lemma 2, the set of solutions $\{V_c : c > 0\}$ of equation (1) is locally bounded in S. Hence, by standard (interior) regularity theory [**6**], their limit U belongs to $C^{2,1}(S)$ and satisfies equation (1). Because the functions V_c are all positive and increasing in c, $U > 0$ in S. Next, since for every $c > 0$,

(12) $$0 \le V_c \le W \quad \text{in } \overline{S} \setminus \{(0,0)\}$$

and $W(x,0) = 0$ for $x \in \mathbf{R}^n \setminus \{0\}$, W is a barrier function at $t = 0$, ensuring that $U \in C(\overline{S} \setminus \{(0,0)\})$ and that $U(x,0) = 0$ for all $x \in \mathbf{R}^n \setminus \{0\}$.

To prove property (v), we observe that, for any $c > 0$, $\int U(x,t)\,dx \ge \int V_c(x,t)\,dx$ and hence

$$\liminf_{t\downarrow 0} \int U(x,t)\,dx \ge \lim_{t\downarrow 0} \int V_c(x,t)\,dx = c,$$

the integrals being taken over $\mathbf{R}^n$. Since c may be chosen arbitrarily large, the result follows. Finally, the symmetry property (vi) follows from the fact that the functions V_c are all endowed with this property.

We know that the function W has all the properties listed for U in Lemma 4, and we expect it is the *only* function with these properties. However, as yet we have not seen a proof.

In the present case, we can deduce from certain scaling properties of the functions V_c that U must be a similarity solution. This is the content of the next lemma.

LEMMA 5. *The function U can be expressed in the form*

$$U(x,t) = t^{-1/(p-1)} g(\eta), \qquad \eta = |x|/t^{1/2}.$$

PROOF. Set $u(x,t) = V_1(x,t)$, and define the family of functions

(13) $$u_k(x,t) = k^{2/(p-1)} u(kx, k^2 t), \qquad k > 0.$$

Then for each $k > 0$, u_k is a solution of the problem

$$\text{(14)} \qquad \begin{aligned} u_t &= \Delta u - u^p \quad \text{in } S, \\ u(x,0) &= k^{2/(p-1)}\delta(kx) \quad \text{in } \mathbf{R}^n. \end{aligned}$$

Observe that the initial condition (14) may be replaced by

$$u(x,0) = k^{2/(p-1)-n}\delta(x).$$

Thus, defining the function $c(k) = k^{2/(p-1)-n}$, we obtain, in view of the uniqueness of V_c,

$$u_k(x,t) = V_{c(k)}(x,t).$$

Note that since $p < (n+2)/n$, $c(k) \to \infty$ as $k \to \infty$, and hence,

$$\text{(15)} \qquad u_k(x,t) \to U(x,t) \quad \text{as } k \to \infty \text{ for } (x,t) \in \overline{S}\backslash\{(0,0)\}.$$

Now let $l > 0$. Then

$$\text{(16)} \qquad u_{kl}(x,t) = (kl)^{2/(p-1)}u(klx,(kl)^2t) = l^{2/(p-1)}u_k(lx,l^2t).$$

Hence, if we let $k \to \infty$ in (16) and we use (15) we obtain

$$\text{(17)} \qquad U(x,t) = l^{1/(p-1)}U(lx,l^2t).$$

Finally, setting $l = t^{-1/2}$, we obtain

$$U(x,t) = t^{-1/(p-1)}U(x/t^{1/2},1) = t^{-1/(p-1)}\hat{U}(|x|/t^{1/2},1)$$

which is the desired form if we set $g(r) = \hat{U}(r,1)$.

It is easily verified that the properties (i)–(vi) of U, listed in Lemma 4, ensure that g is a solution of Problem (P). By the uniqueness theorem for this problem [**3**] we may conclude that $g = f$ and hence that $U = W$. This proves Theorem 1.

REMARK. Note that the proof of Lemma 5 hinges on two invariance properties of U:

(a) invariance with respect to scaling: see (17);

(b) invariance with respect to rotations: see Lemma 4(vi).

The fact that we have a uniqueness theorem for precisely this class of functions enables us to state that $U = W$.

Using the scaling method of [**5**], we can deduce from (13) the limiting behaviour of $V_c(x,t)$ as $t \to \infty$. For if we set $t = 1$ and let $k \to \infty$ in (13) we obtain

$$\lim_{k\to\infty} k^{2/(p-1)}V_c(kx,k^2) = U(x,1)$$

uniformly on bounded sets in $\mathbf{R}^n$. Thus, with $k^2 = s$ and $kx = y$, we find that

$$\lim_{s\to\infty} s^{1/(p-1)}V_c(y,s) = U(y/s^{1/2},1) = f(|y|/s^{1/2}),$$

uniformly on sets $P_a = \{(y,s) \colon |y| \leq as^{1/2}\}$, $a > 0$. This proves Theorem 2.

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