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ON THE ELLIPTIC EQUATION $D_i[a_{ij}(x)D_jU] - k(x)U + K(x)U^p = 0$

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ABSTRACT. The problem of the existence and nonexistence of entire, positive solutions to the uniformly elliptic, semilinear equation $D_i[a_{ij}(x)D_jU] - k(x)U + K(x)U^p = 0$ in $\mathbf{R}^n$, where $p > 1$, is studied. A limiting case when $K(x)$ is negative and has quadratic decay at infinity is also treated.

I. Introduction. The problem of conformal deformation of metric with prescibed scalar curvature for a class of simple Riemannian manifold leads to, naturally, the study of the following more general uniformly elliptic, semilinear equation:

$$LU - k(x)U + K(x)U^p = 0 \quad \text{in } \mathbf{R}^n, \tag{1.1}$$

where $n \geq 3$, $p > 1$, and $L = D_i[a_{ij}(x)D_j(\ )]$, and the functions $a_{ij} = a_{ji}$, for $i, j = 1, \ldots, n$, are measurable and satisfy the uniform ellipticity condition

$$\lambda^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2 \tag{1.2}$$

for all $x, \xi \in \mathbf{R}^n$ with $\lambda \geq 1$ being a fixed constant.

The existence and nonexistence of positive solutions to (1.1) was studied extensively in **[N]** for the special case that $a_{ij} = \delta_{ij}$, and some of the main results have been extended to (1.1) by C. Kenig and W. M. Ni **[KN1, KN2]**. The main existence result in **[KN1]** may be described roughly as follows: *If* $0 \leq k(x) \leq c(1 + |x|)^{2+\varepsilon}$, *and* $|K(x)| \leq c(1+|x|)^{2+\varepsilon}$, *for some positive constants* c, ε, *then* (1.1) *has infinitely many bounded solutions in* $\mathbf{R}^n$ *with positive lower bounds.* It is also shown in **[KN1]** and **[N]** that if K is negative and $|K(x)| \geq c(1+|x|)^{\varepsilon-2}$ at ∞ for some c, ε positive constants, then (1.1) has no positive solutions in $\mathbf{R}^n$ provided $k(x) \geq 0$. Thus it remains an open question for the limiting case when $K(x)$ is negative and has quadratic decay at ∞.

In this work, we settle this question. Our main result, which can be deduced from Theorem 3.4 in §3, is the following

THEOREM. *If* $k(x) \geq 0$ *and* $K(x)$ *is bounded and satisfies* $K(x) \leq -c(1+|x|)^{-2}$ *for some constant* $c > 0$, *then* (1.1) *has no positive solution in* $\mathbf{R}^n$.

We also prove a slightly different version of the existence result without the positivity hypothesis on k in §2. A few remarks and some generalizations are discussed in §4.

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II. Existence results. We first consider the linear equation

$$D_i[a_{ij}(x)D_jU] = f(x) \quad \text{in } \mathbf{R}^n, \tag{2.1}$$

where (a_{ij}) satisfies (1.2) and $f \in L^\infty_{\text{loc}}(\mathbf{R}^n)$.

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PROPOSITION 2.1. *For any $\varepsilon \in R$, (2.1) has a unique solution $U \in C^\alpha(\mathbf{R}^n) \cap H^1_{\mathrm{loc}}(\mathbf{R}^n)$ for which $\lim_{|x|\to\infty} U(x) = \varepsilon$, provided that $|f(x)| \leq (1+|x|^2)^{-1} w(|x|)$ with $\int_1^\infty \gamma^{-1} w(r)\,dr < \infty$.*

PROOF. Let $G(x,y)$ be the Green's function of operator L, $L = D_i[a_{ij}D_j(\ )]$. Then by the estimate $0 \leq G(x,y) \leq K_1|x-y|^{2-n}$ where $K_1 = K_1(n,\lambda)$, cf. [**LSW**], we see that $U(x) = \varepsilon - \int_{\mathbf{R}^n} G(x,y) f(y)\,dy$ is the unique solution of (2.1), and that $\lim_{|x|\to\infty} U(x) = \varepsilon$ and that $U \in C^\alpha(\mathbf{R}^n) \cap H^1_{\mathrm{loc}}(\mathbf{R}^n)$, for some $\alpha > 0$. Q.E.D.

We also need the following result [**N**, Theorem 2.10]:

PROPOSITION 2.2. *Suppose $\psi \geq \varphi$ are respectively an entire supersolution and subsolution of*

$$D_i[a_{ij}(x) D_j U] + f(x,U) = 0, \tag{2.2}$$

where (a_{ij}) as in (1.2) and $f(x,y)$ is a locally Hölder continuous function which is locally lipschitz in y. Then there is an entire solution U of (2.2) such that $\varphi \leq U \leq \psi$ in $\mathbf{R}^n$.

Now we consider (1.1).

THEOREM 2.3. *Let $w\colon \mathbf{R}^+ \to \mathbf{R}^+$ be a locally bounded function with*

$$\int_1^\infty r^{-1} w(r)\,dr \equiv A < \infty.$$

Then there exists a positive constant $\theta = \theta(\lambda, n, A)$ such that if $|K(x)| \leq C_0(1+|x|)^{-2} w(|x|)$ and $|k(x)| \leq \theta(1+|x|)^{-2} w(|x|)$, for some $C_0 > 0$ constant, then (1.1) has a family of positive solutions in $C^\alpha(\mathbf{R}^n) \cap H^1_{\mathrm{loc}}(\mathbf{R}^n)$. Moreover, each of these tends to some positive limit at infinity.

PROOF. First we claim that there exists a $\theta_1 > 0$, such that

(i) the unique solution of $D_i[a_{ij}(x)D_jU_\varepsilon] - C(1+|x|)^{-2}w(|x|) = 0$ with $\lim_{|x|\to\infty} U_\varepsilon(x) = \varepsilon$, which is guaranteed by Proposition 2.1, satisfies $0 < U_\varepsilon \leq \varepsilon$ for any $C \in (0,\theta_1)$ and $\varepsilon \in (1/3, 1/2)$,

(ii) the unique solution of $D_i[a_{ij}(x)D_jU^\varepsilon] + C(1+|x|)^{-2}w(|x|) = 0$ with $\lim_{|x|\to\infty} U^\varepsilon(x) = \varepsilon$, which is guaranteed by Proposition 2.1, satisfies $\varepsilon \leq U^\varepsilon < 1$ for any $C \in (0,\theta_1)$ and $\varepsilon \in (1/3, 1/2)$.

These are easy consequences of the representation formulas for the solutions. Here the constant θ_1 depends only on λ, n and A.

Now we choose $\theta = \theta_1/3$. By changing the dependent variable if necessary, we can, since $p > 1$, assume $|K(x)| \leq \theta(1+|x|)^{-2}w(|x|)$. Choosing $C = 2\theta_1/3$ in (i) and (ii) above, we have, for any $\varepsilon \in (1/3,1/2)$, functions $0 < U_\varepsilon \leq U^\varepsilon < 1$ such that

$$\begin{aligned}
D_i[a_{ij}(x)D_jU^\varepsilon] &- k(x)U^\varepsilon + K(x)U^{\varepsilon^p} \\
&= -2\theta_1 w(|x|)/3(1+|x|)^2 - k(x)U^\varepsilon + K(x)U^{\varepsilon^p} \\
&\leq -2\theta_1 w(|x|)/3(1+|x|)^2 + |K(x)| + |k(x)| \leq 0,
\end{aligned}$$

and, similarly, $D_i[a_{ij}(x)D_jU_\varepsilon] - k(x)U_\varepsilon + K(x)U_\varepsilon^p \geq 0$.

Now applying Proposition 2.2 we find a solution U of (1.1) with $0 < U_\varepsilon \leq U \leq U^\varepsilon < 1$ and $\lim_{|x|\to\infty} U(x) = \varepsilon$. Q.E.D.

III. Nonexistence results. We again begin with (2.1). The following estimate is of independent interest:

PROPOSITION 3.1. *Let U be an entire solution of* (2.1) *with $U(0) \geq 0$, and let $M(R) = \max_{|x|=R} U(x)$. Then, for all $R > 0$, we have*

$$M(R) \geq c(n,\lambda) \int_0^{2R/3} r^{-1} w(r)\, dr \tag{3.0}$$

provided that $f(x) \geq (1+|x|^2)^{-1} w(|x|) \geq 0$. If U is a positive solution on $\{x\colon |x| \geq R_0\}$ for some $R_0 > 0$, and the integral $\int_{R_0}^\infty r^{-1} w(r)\, dr = \infty$, then (3.0) *remains true for all suitably large R.*

PROOF. To show the first part of the proposition, we solve

$$\begin{aligned} LU_R &= f \qquad \text{in } B_R = \{x \in \mathbf{R}^n\colon |x| < R\}, \\ U_R &= M(R) \quad \text{on } \partial B_R. \end{aligned}$$

By the maximum principle and the representation formulas for U_R we have that

$$\begin{aligned} M(R/2) &= \max_{|x|=R/2} U(x) \\ &\leq \max_{|x|=R/2} U_R(x) = M(R) - \int_{|y|<R} G_R(\overline{x}, y) f(y)\, dy \end{aligned}$$

where $|\overline{x}| = R/2$ and G_R is the Green's function of L on B_R. Thus,

$$\begin{aligned} M(R) - M(R/2) &\geq c(n,\lambda) \int_{R/3<|y|<2R/3} f(y) \left|\overline{x} - y\right|^{2-n}\, dy \\ &\geq c(n,\lambda) \int_{R/3}^{2R/3} r^{-1} w(r)\, dr, \end{aligned}$$

and (3.0) follows.

Next we let $U > 0$ be a solution of (2.1) on $\mathbf{R}^n - B_{R_0}$, and consider the problem

$$\begin{aligned} LV_R &= f \qquad \text{in } B_R - \overline{B}_{R_0}, \\ V_R &= \overline{M}(R) \quad \text{for } |x| = R_0 \text{ or } R, \end{aligned}$$

where $\overline{M}(R) = \max\{M(R_0), M(R)\}$. Then for $R > R_0$ we have

$$U(x) \leq V_R(x) = \overline{M}(R) - \int_{R_0<|y|<R} G_{R_0,R}(x,y) f(y)\, dy,$$

where $G_{R_0,R}$ is the Green's function of L on $B_R - B_{R_0}$, $x \in B_R - B_{R_0}$. In particular, since $U > 0$, we have

$$\overline{M}(R) \geq \int_{R_0<|y|<R} G_{R_0,R}(x,y) f(y)\, dy$$

for all $R > R_0$ and $x \in B_R - B_{R_0}$. Letting $R \to \infty$, $G_{R_0,R}(x,y)$ converges monotonely to $G_{R_0,\infty}(x,y) \geq c(n,\lambda)|x-y|^{2-n}$. By the hypothesis $\int_{R_0}^\infty r^{-1} w(r)\, dr =$

∞, we conclude that $\overline{M}(R) \to \infty$ as $R \to \infty$. Thus we can assume, for some constant $R_1 \geq R_0$, that $\overline{M}(R) = M(R)$ for $R \geq R_1$. Now,

$$M(R/2) = \max_{|x|=R/2} U(x) \leq \max_{|x|=R/2} V_R(x)$$
$$\leq \overline{M}(R) - c(n,\lambda)\int_{R/3}^{2R/3} r^{-1}w(r)\,dr = M(R) - c(n,\lambda)\int_{R/3}^{2R/3} r^{-1}w(r)\,dr$$

for $R \geq R_1$, and this implies

$$M(R) \geq M(R_1) + c(n,\lambda)\int_{R_1}^{2R/3} r^{-1}w(r)\,dr. \quad \text{Q.E.D.}$$

LEMMA 3.2. *Let U be a positive supersolution of*

$$LU + K(x)U^p = 0 \quad \textit{in } \mathbf{R}^n. \tag{3.1}$$

Then $\Delta\overline{U} + C_1K(x)\overline{U}^p \leq 0$ has a positive solution $\overline{U}$ provided $K(x) \geq 0$ and $K(x) \not\equiv 0$ in $\mathbf{R}^n$, where $C_1 = C_1(n,\lambda)$, $p > 1$.

COROLLARY 3.3. *A consequence of Lemma 3.2 is the following: If $K(x) \geq 0$ in $\mathbf{R}^n$ ($\mathbf{R}^n/B_R$ respectively), and $\overline{K}_p(r) \geq Cr^l$ for r large, where $C > 0$ constant, $l \geq (n-2)(p-1)-2$. Then* (3.1) *does not possess any positive supersolution in $\mathbf{R}^n$ ($\mathbf{R}^n/B_R$ respectively).*

For the definition of $\overline{K}_p(r)$ and the proof, see [**N**].

PROOF OF LEMMA 3.2. For $R \in [1,\infty)$, let

$$V_R(x) = \int_{B_R} G_R(x,y)K(y)U^p(y)\,dy,$$

where G_R is the Green's function of L on the ball $B_R(0)$. By the maximal principle, $V_R \leq U$ on B_R. If we let $\overline{G}_R$ be the Green's function of Δ on B_R, then it is not hard to see that there exists a constant $C_1 = C_1(n,\lambda)$ such that

$$U(x) \geq V_R(x) \geq \overline{V}_R(x) = C_1(n,\lambda)\int_{B_{2R/3}} \overline{G}_R(x,y)K(y)U^p(y)\,dy.$$

So we have a family $\{\overline{V}_R\}$ such that

(i) $0 \leq \overline{V}_R(x) \leq U(x)$ for $x \in B_{2R/3}$,

(ii) $\Delta\overline{V}_R = -C_1KU^p \leq -C_1K(x)\overline{V}_R^p$ on $B_{2R/3}$,

(iii) for each R_0 sufficiently large, there is a number $\varepsilon(R_0) > 0$ such that $\overline{V}_R(x) \geq \varepsilon(R_0)$ for $x \in B_{R_0/2}$ and $R > 2R_0$.

The last statement follows from $K \geq 0$, that $K \not\equiv 0$ in $\mathbf{R}^n$, that $U(x) > 0$ in $\mathbf{R}^n$ and that $\overline{G}_{R'}(x,y) \leq \overline{G}_{R''}(x,y)$ whenever $R' \leq R''$ and $x, y \in B_{R'}$.

Then there exists a sequence $R_m \to \infty$, such that $\overline{V}_{R_m} \to \overline{V}_0$ in $H^1_{\text{loc}}(\mathbf{R}^n)$ and in the appropriate local Hölder norm. Thus $\overline{V}_0 \in H^1_{\text{loc}}(\mathbf{R}^n) \cap C^\alpha(\mathbf{R}^n)$ and satisfies

(i) $\overline{V}_0(x) \leq U(x)$ on $\mathbf{R}^n$,

(ii) $\overline{V}_0(x) \geq \varepsilon(R_0)$ for $x \in B_{R_0/2}$,

(iii) $\Delta\overline{V}_0 \leq -C_1K(x)\overline{V}_0^p$, in fact, $\Delta\overline{V}_0 = -C_1K(x)U^p$. Q.E.D.

Now we treat the case $K(x) \leq 0$. Our main result is the following theorem.

THEOREM 3.4. *There is no positive subsolution to* (3.1) *provided* $K(x) \le -A(1+|x|)^{-2}$, *for some constant* $A > 0$.

In proving this theorem, it seems more convenient to consider the following inequality:

$$LU \ge KU^p \quad \text{in } \mathbf{R}^n, \tag{(3.1)$'$}$$

where $K \ge A/(1+|x|^2)$ and prove that (3.1)$'$ has no entire positive solution. (Note that we have changed the sign of K here.)

We first make a few observations.

(a) *A differential inequality.* If $U > 0$ is a solution of (3.1)$'$, then $\int_{B_r} ULU\,dx \ge \int_{B_r} KU^{p+1}\,dx$. This implies

$$\int_{\partial B_r} U(a_{ij}D_jU)\nu_i\,ds \ge \int_{B_r} KU^{p+1}\,dx + \int_{B_r} a_{ij}D_iUD_jU\,dx$$

and so

$$C_0\int_{\partial B_r} U|DU|\,ds \ge \int_{B_r} KU^{p+1}\,dx + \int_{B_r} |Du|^2\,dx,$$

for some positive constant $C_0 = C_0(n,\lambda)$. We integrate the above inequality from $r=0$ to $r=R$ to find that

$$C_0\int_{B_R} U|DU|\,dx \ge \int_0^R\int_{B_r} KU^{p+1}\,dx\,dr + \int_0^R\int_{B_r} |DU|^2\,dx\,dr.$$

(b) *Moser's subsolution estimate.* If $U > 0$ satisfies $LU \ge 0$ in B_{4R}, then

$$M(R)^l \le C(n,\lambda,l)\fint_{B_{2R}} U^l\,dx,$$

where $l > 1$, $M(R) = \sup_{|x|\le R} U(x)$, and

$$\fint_{B_{2R}} U^l\,dx = \frac{1}{|B_{2R}|}\int_{B_{2R}} U^l\,dx.$$

(c) *Exponential growth estimate.* Let U be a positive subsolution of (3.1)$'$ in $\mathbf{R}^n$. Then there exists α, C^* two positive constants such that $M(R) \ge \exp(C^*R^\alpha)$.

PROOF. Let $V_R(x) = M(2R) - \int_{B_{2R}} G_{2R}(x,y)K(y)U^p(y)\,dy$. Then $0 < U \le V_R$ on B_{2R}. Thus

$$\begin{aligned} M(R/2) &= \sup_{|x|\le R/2} U(x) \\ &\le \sup_{|x|\le R/2} V_R(x) = M(2R) - \int_{B_{2R}} G_{2R}(\overline{x},y)\,U^p(y)K(y)\,dy \end{aligned}$$

for some $\overline{x}$, $|\overline{x}| = R/2$. Thus for some constant $C_0 = C_0(n,\lambda)$, we have

$$\begin{aligned} M(2R) &\ge M(R/2) + C_0\int_{B_{3R/2}} \overline{G}_{2R}(\overline{x},y)\,U^p(y)K(y)\,dy \\ &\ge M(R/2) + C_0A\fint_{B_R} U^p(y)\,dy \ge M(R/2) + C_1(n,\lambda,p,A)M^p(R/2). \end{aligned}$$

Without loss of generality we assume $U(0) = 1$. Thus, we have $M(2R) \geq (1+C_1)M(R/2)$, and $M(R) \geq 1$ for all $R > 0$. Hence, $M(R) \geq C_2 R^{\alpha_1}$ for some $\alpha_1 > 0$, $C_2 > 0$.

Next we go back to the inequality $M(2R) \geq M(R/2) + C_1 M^p(R/2)$, and iterate to obtain

$$M(4^m R_0) \geq C_1 M^p(4^{m-1}R) \geq \cdots \geq C_1^{\gamma_m} M^{p^m}(R_0),$$

$\gamma_m = 1 + p + \cdots + p^{m-1} = (p^m - 1)/(p-1)$. Hence,

$$\log M(4^m R_0) \geq p^m[\log M(R_0) + (\gamma_m/p^m)\log C_1].$$

Taking R_0 large enough so that $\log M(R_0) + (\gamma_m/p^m)\log C_1 > 1$, we find that $M(4^m R_0) \geq \exp(p^m) \geq \exp[C^*(R_0 4^m)^\alpha]$. Q.E.D.

Proof of Theorem 3.4. By (a),

$$\int_{B_R} U|DU|\,dx \geq C_0\left[\int_0^R\int_{B_r}|DU|^2\,dx + \int_0^R\int_{B_r}KU^{p+1}\,dx\,dr\right].$$

Since $p > 1$, we may choose $1 < s_0 < 2$ sufficiently close to 1 so that $2s_0/(2-s_0) < p+1$. Thus,

$$\begin{aligned}\int_{B_r} U^{s_0}|DU|^{s_0}\,dx &\geq CR^{-n(s_0-1)}\left(\int_{B_R}U|DU|\right)^{s_0}\\ &\geq CR^{-n(s_0-1)}\left[\int_0^R\int_{B_R}(|DU|^2+KU^{p+1})\,dx\,dr\right]^{s_0}\\ &\geq C_0\left[\int_0^R\int_{B_r}(|DU|^2+KU^{p+1})\,dx\,dr\right]^{s_1}\end{aligned}$$

for $s_1 = (1+s_0)/2 > 1$ and for R large. Here we have used the fact that $K(x) \geq A(1+|x|)^{-2}$ and $\int_{B_R} U^{p+1}\,dx$ has exponential growth. Since $U^{s_0}|DU|^{s_0} \leq (s_0/2)|DU|^2 + ((2-s_0)/2)U^{2s_0/(2-s_0)}$,

$$\int_{B_R}\left(|DU|^2 + U^{2s_0/(2-s_0)}\right)\,dx \geq C_1\left[\int_0^R\int_{B_r}(|DU|^2+KU^{p+1})\,dr\right]^{s_1}.$$

Finally, we get

$$\begin{aligned}\int_{B_R}(|DU|^2+KU^{p+1})\,dx &\geq \int_{B_R}\left(|DU|^2+U^{2s_0/(2-s_0)}\right)\,dx\\ &\geq C_1\left[\int_0^R\int_{B_r}(|DU|^2+KU^{p+1})\,dx\,dr\right]^{s_1}\end{aligned}$$

for R large. This implies that $f'(R) \geq C_1 f(R)^{s_1}$ for R large, where

$$f(R) = \int_0^R\int_{B_r}(|DU|^2+KU^{p+1})\,dx\,dr;$$

hence if we choose R_0 a fixed large number, then

$$(f(R_0)^{1-s_1} - f(R)^{1-s_1})/(s_1-1) \geq C_1(R-R_0) \quad \text{for } R > R_0.$$

As $R \to \infty$, $f(R) \to \infty$, a contradiction. Q.E.D.

IV. Final remarks. (i) When $p \geq 1$, the proof of Theorem 3.4 yields the following result: If $U > 0$ satisfies $LU \geq K(x)U^p$ in $\mathbf{R}^n$, where $K(x) \geq (1+|x|)^{-2}w(|x|) \geq 0$, with $\int_1^\infty r^{-1}w(r)\,dr = \infty$, then U cannot be bounded above.

(ii) In case $p = 1$, (1.1) reduces to the linear equation

$$LU + V(x)U = 0. \tag{4.1}$$

It is standard [**KN1**] to prove that if $V(x) \in L^q_{\mathrm{loc}}(\mathbf{R}^n)$ with $V \leq 0$ on $\mathbf{R}^n$ and $q > n/2$, then (4.1) always admits entire positive solutions. This is Theorem 3.4 is not true when $p = 1$.

We also notice that [**B**] there is no nontrivial solution $U \in L^p_{\mathrm{loc}}(\mathbf{R}^n)$ to $\Delta U - |U|^{p-1}U = 0$ in $D'(\mathbf{R}^n)$ for $p > 1$.

(iii) It is easy to show that there exist two positive constants $C_1 = C_1(n, \lambda)$, $C_2 = C_2(n, \lambda)$ so that: If $|V(x)| \leq C_1/(1+|x|^2)$, then (4.1) admits positive entire solutions. If $V(x) \geq C_2/(1+|x|^2)$, then (4.1) does not have any positive solution on $\mathbf{R}^n$ or even on a bounded ball.

(iv) The local existence and nonexistence of positive solutions of (1.1) still remain an interesting question in general. We have learned in personal communication with Dr. W. Y. Ding that, a simple choice of functions a_{ij} on the unit ball B, the uniformly elliptic, semilinear equation

$$\begin{cases} LU + U^{(n+2)/(n-2)} = 0 & \text{in } B \\ U = 0 & \text{on } \partial B \end{cases}$$

may have a positive solution.

(v) In the case $(a_{ij}(x))$ is a symmetric, positive definite matrix with measurable entries and of which the eigenvalues are of order of magnitude $|x|^{a(2-n)}$ at infinity, with $-\infty < a < 1$, our results remain true with obvious modifications. More precisely, we have the following

THEOREM 2.3$'$. *Let $w\colon \mathbf{R}^+ \to \mathbf{R}^+$ be locally bounded with $\int_1^\infty r^{-1}w(r)\,dr = A < \infty$. Then there exists a positive constant $\theta = \theta(A, L)$ such that if $|K(x)| \leq C(1+|x|)^{a(2-n)-2}w(|x|)$, for some $C > 0$ constant, and*

$$|k(x)| \leq \theta(1+|x|)^{a(2-n)-2}w(|x|),$$

then (1.1), with a_{ij}'s above, has a family of positive solutions in $C^\alpha(\mathbf{R}^n) \cap H^1_{\mathrm{loc}}(\mathbf{R}^n)$. Moreover, each of these tends to some positive limit at infinity.

THEOREM 3.4$'$. *There is no positive subsolution to (3.1), provided $K(x) \leq -C(|x|+1)^{a(2-n)-2}$, for some constant $C > 0$, where a_{ij}'s is in this remark.*

The proofs of the above theorems are similar to those which have been carried out when L is the uniform elliptic operator. The estimates we needed are available in [**FKS**] and [**FJE**].

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