COMPACTNESS IN $L^2$ AND THE FOURIER TRANSFORM

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Abstract. The Riesz-Tamarkin compactness theorem in $L^p(\mathbb{R}^n)$ employs notions of $L^p$-equicontinuity and uniform $L^p$-decay at $\infty$. When $1 < p < 2$, we show that these notions correspond under the Fourier transform, and establish new necessary and sufficient criteria for compactness in $L^2(\mathbb{R}^n)$.

An oft-quoted classical result characterizing compact sets in $L^p(\mathbb{R}^n)$ is due to M. Riesz and J. D. Tamarkin (see [1, 2, 4]):

Theorem. A bounded subset $K$ of $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, is conditionally compact if and only if

(I) $\int_{\mathbb{R}^n} |f(x + y) - f(x)|^p \, dx \to 0$ as $y \to 0$ uniformly for $f$ in $K$, and

(II) $\int_{|x| > R} |f(x)|^p \, dx \to 0$ as $R \to \infty$ uniformly for $f$ in $K$.

Property (I) is a uniform smoothness property. By analogy with the terminology of Arzela-Ascoli, we say the functions in $K$ are $L^p$-equicontinuous if (I) holds. Property (II) is a uniform decay property. The connection between smoothness and decay through the Fourier transform has been well explored [6]. Yet the following nice equivalence seems to be new:

Theorem 1. Let $K$ be a bounded subset of $L^2(\mathbb{R}^n)$ and let $\hat{K}$ be the Fourier transform of $K$, $\hat{K} = \{ \hat{f} | f \in K \}$. The functions of $K$ are $L^2$-equicontinuous if and only if the functions of $\hat{K}$ decay uniformly in $L^2$, and vice versa. That is, $K$ satisfies (I) in $L^2$ if and only if $\hat{K}$ satisfies (II) in $L^2$, and vice versa.

Combining this result with the Riesz-Tamarkin theorem, we obtain two alternative characterizations of compact sets in $L^2(\mathbb{R}^n)$:

Theorem 2. A bounded subset $K$ of $L^2(\mathbb{R}^n)$ is conditionally compact if and only if

\[ \int |f(x + y) - f(x)|^2 \, dx \to 0 \text{ as } y \to 0, \text{ and } \int |\hat{f}(\xi + \omega) - \hat{f}(\xi)|^2 \, d\xi \to 0 \text{ as } \omega \to 0, \]

both uniformly for $f$ in $K$.

Theorem 3. A bounded subset $K$ of $L^2(\mathbb{R}^n)$ is conditionally compact if and only if

\[ \int_{|x| > R} |f(x)|^2 \, dx \to 0 \text{ and } \int_{|\xi| > R} |\hat{f}(\xi)|^2 \, d\xi \to 0 \text{ as } R \to \infty, \]

both uniformly for $f$ in $K$. 

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Theorem 1 is an easy consequence of the theorem below, which offers some results in \(L^p\), \(1 \leq p \leq 2\).

**Theorem 4.** Let \(K\) be a bounded subset of \(L^p\), \(1 \leq p \leq 2\). If \(K\) satisfies (I) (resp. (II)) in \(L^p\), then \(\hat{K}\) satisfies (II) (resp. (I)) in \(L^q\), where \(1/p + 1/q = 1\). (If \(q = \infty\), conditions (I) and (II) are to be stated in the obvious way using the sup norm.)

Let us set our notation and recall some basic results. For \(f \in L^1(\mathbb{R}^n)\),
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} \, dx.
\]
Recall [3]:

1. The Fourier transform above extends to a bounded linear map \(f \to \hat{f}\) from \(L^p\) to \(L^q\), for \(1 \leq p \leq 2\) and \(1/p + 1/q = 1\), so \(\|\hat{f}\|_q \leq C_p \|f\|_p\) for \(f \in L^p\).
2. For \(f \in L^p\), \(\omega \in \mathbb{R}^n\), we have \([e^{-i\omega \cdot f(x)}]\hat{f}(\xi) = \hat{f}(\xi + \omega)\) in \(L^q\).
3. For \(f \in L^p\), \(\psi\) in the Schwartz class \(\mathcal{S}\), \((f * \psi)(\xi) = \hat{f}(\xi) \hat{\psi}(\xi)\) in \(L^q\), where \(f * \psi(x) = \int_{\mathbb{R}^n} f(x - y) \psi(y) \, dy\).

**Proof of Theorem 4.** First, we assume \(K\) satisfies (II) in \(L^p\). Let \(M\) be a bound for \(K\) in \(L^p\). For \(f \in K\),
\[
\hat{f}(\xi + \omega) - \hat{f}(\xi) = \left[ (e^{-i\omega \cdot x} - 1) f(x) \right] \hat{f}(\xi),
\]
whence
\[
\|\hat{f}(\xi + \omega) - \hat{f}(\xi)\|_q \leq C_p \| (e^{-i\omega \cdot x} - 1) f(x) \|_p
\]
\[
\leq C_p \left( \int_{|x| < R} (|\omega| |x| |f(x)|)^p \, dx + 2 \int_{|x| > R} |f(x)|^p \, dx \right)^{1/p}.
\]
Let \(\varepsilon > 0\). Because of (II) we may choose \(R\) so large that the second term here is less than \(\frac{1}{2} (\varepsilon/C_p)^p\) independent of \(f\) in \(K\). Then since \(\int_{|x| < R} (|\omega| |f(x)|)^p \, dx \leq (RM)^p\) for \(f \in K\), we have \(\|\hat{f}(\xi + \omega) - \hat{f}(\xi)\|_q < \varepsilon\) if \(\omega\) is sufficiently small, \(|\omega|^p < \frac{1}{2} (\varepsilon/C_p RM)^p\), independent of \(f\) in \(K\). So \(\hat{K}\) satisfies (I) in \(L^q\).

Now assume \(K\) satisfies (I) in \(L^p\). We seek to show that functions in \(\hat{K}\) decay uniformly in \(L^q\). Let \(\psi(x) = (2\pi)^{-n/2} e^{-|x|^2/2}, \psi_R(x) = \psi(Rx) R^n\), so that \(\psi_R \psi_R(\xi/R) \in \mathcal{S}^n\), with \(\hat{\psi}(\xi) = e^{-|\xi|^2/2}, \hat{\psi}_R(0) = \hat{\psi}(R) = 1\). Now for \(|\xi| \geq 2R, \frac{1}{2} \leq 1 - \hat{\psi}_R(\xi), \) so for \(f \in K\),
\[
\frac{1}{2} \left( \int_{|\xi| > 2R} |\hat{f}(\xi)|^q \, d\xi \right)^{1/q} \leq \|\hat{f}(\xi)(1 - \hat{\psi}_R(\xi))\|_q
\]
\[
\leq C_p \|f(x) - f * \psi_R(x)\|_p
\]
\[
= C_p \left( \int \left[ \int (f(x) - f(x - y)) \psi_R(y) \, dy \right]^p \, dx \right)^{1/p}.
\]
By Jensen's inequality and Fubini's theorem, this is
\[
\leq C_p \left[ \int \left[ \int (f(x) - f(x - y))^p \, dx \right] \psi(y) \, dy \right]^{1/p}.
\]
Now define a uniform $L^p$ modulus of continuity for $K$,

$$H(y) = \sup_{f \in K} \int |f(x) - f(x - y)|^p \, dx.$$ 

By (I), $H(y) \to 0$ as $y \to 0$, and $H(y) \leq (2M)^p$ for all $y$. From above, we have

$$\left[ \int_{|\xi| > \frac{R}{2}} |\hat{f}(\xi)|^q \, d\xi \right]^{1/q} \leq 2C_p \left[ \int H\left( \frac{y}{R} \right) \psi(y) \, dy \right]^{1/p} \to 0$$

as $R \to \infty$ uniformly for $f$ in $K$. Hence, $\hat{K}$ satisfies (II).

We conclude with a small application, which illustrates a principle known in information theory (see [5]) that an operator in $L^2$ that is “band limited and time limited” is compact.

Fix any $\phi_1(x), \phi_2(x)$ bounded functions on $\mathbb{R}^n$ which satisfy $\lim_{|x| \to \infty} \phi_i(x) = 0$, $i = 1, 2$, and let $\phi_i$ denote the multiplication operator on $L^2$ given by $u(x) \to \phi_i(x)u(x)$, $i = 1, 2$. Let $F$ denote the Fourier transform operator $u \to Fu = \hat{u}$. Define an operator $T$ on $L^2$ by $T = \phi_1 F \phi_2$. Assume $\phi_1(x)$ is continuous.

**Proposition.** $T$ is a compact operator on $L^2$.

**Proof.** Let $K$ be a bounded set in $L^2$. Clearly, the set $\phi_2 K$ has the uniform decay property (II) in $L^2$. From Theorem 1, the set $F \phi_2 K$ is $L^2$-equicontinuous (has property (I)). The set $TK = \phi_1 F \phi_2$ is also $L^2$-equicontinuous, and also has the uniform decay property (II). By Riesz-Tamarkin, it follows that $TK$ is precompact. Q.E.D.

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**References**