PETTIS INTEGRABILITY AND THE EQUALITY OF THE NORMS OF THE WEAK* INTEGRAL AND THE DUNFORD INTEGRAL

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Abstract. If \((\Omega, \Sigma, \mu)\) is a perfect finite measure space and \(X\) is a Banach space, then it is shown that \(X^*\) has the \(\mu\)-Pettis Integral Property if and only if
\[
\left\| \left(\text{weak*}\right) - \int_\Omega f \, d\mu \right\| = \left\| \left(\text{Dunford}\right) - \int_\Omega f \, d\mu \right\|
\]
for every bounded weakly measurable function \(f: \Omega \to X^*\).

1. Introduction. Let \(X\) be a Banach space with dual \(X^*\) and \((\Omega, \Sigma, \mu)\) a finite measure space. If \(f: \Omega \to X^*\) is bounded and weakly measurable, that is if \(x^{**} \circ f\) is measurable for every \(x^{**} \in X^{**}\), then it can easily be shown that

(i) for every \(E \in \Sigma\), there exists \(x_E^* \in X^*\) such that, for every \(x \in X\),
\[
x_E^*(x) = \int_E x \circ f \, d\mu
\]
and

(ii) for every \(E \in \Sigma\), there exists \(x_E^{**} \in X^{**}\) such that, for every \(x^{**} \in X^{**}\),
\[
x_E^{**}(x^{**}) = \int_E x^{**} \circ f \, d\mu.
\]
The element \(x_E^*\) is called the weak* integral of \(f\) over \(E\), denoted by \((w^*) - \int_E f \, d\mu\), and \(x_E^{**}\) is called the Dunford integral of \(f\) over \(E\), denoted \((D) - \int_E f \, d\mu\). By definition, \(f\) is Pettis integrable if and only if \((D) - \int_E f \, d\mu \in X^*\).

A Banach space \(Y\) is said to have the \(\mu\)-Pettis Integral Property, or \(\mu\)-PIP if every bounded weakly measurable function \(f: \Omega \to Y\) is Pettis integrable. Characterizations and properties of Pettis integrable functions, spaces with the \(\mu\)-PIP, and integration of universally weakly measurable functions can be found in [1, 5, 6, 8-14, 16]. Clearly, \(X^*\) has the \(\mu\)-PIP if and only if for every \(f: \Omega \to X^*\) that is bounded and weakly measurable, \((w^*) - \int_E f \, d\mu = (D) - \int_E f \, d\mu\) for every \(E \in \Sigma\). We show that in fact if \((\Omega, \Sigma, \mu)\) is perfect, then \(X^*\) has \(\mu\)-PIP if and only if \(\|(w^*) - \int_\Omega f \, d\mu\| = \|(D) - \int_\Omega f \, d\mu\|\) for every such function \(f\).

2. Preliminary results. If \(A\) is a finite subset of a Banach space \(X\) and \(\epsilon > 0\), then we define
\[
C_{A, \epsilon} = \left\{ x^* \in B^*: \left| x^*(x) \right| < \epsilon \text{ for every } x \in A \right\},
\]
where \(B^* = \left\{ x^* \in X^*: \|x^*\| \leq 1 \right\}\).
Lemma 1. Let $X$ be a Banach space and $x^{**} \in X^{**}$. Suppose for every $\eta > 0$ there exists a finite subset $A$ of $X^*$ and $\varepsilon > 0$ such that if $x^* \in C_{A, \varepsilon}$, then $|x^{**}(x^*)| < \eta$. Then $x^{**} \in X$.

Proof. Let $x^*_a$ be a net in $\frac{1}{2} B^*$ such that $x^*_a$ converges weak* to $x^*$. Then $(x^*_a - x^*) \in B^*$ and is eventually in $C_{A, \varepsilon}$ for every $A, \varepsilon$. Hence, $x^{**}(x^*_a - x^*)$ converges to zero. Consequently, $x^{**}$ is weak* continuous on $B^*$, and hence, $x^{**} \in X$. □

A function $f$ from a measure space $(\Omega, \Sigma, \mu)$ to a dual Banach space $X^*$ is weak* measurable if $\hat{x} \circ f$ is measurable for every $x \in X$. If $f$ is weak* measurable we define the Pettis norm of $f$ by

$$\|f\|_P = \sup_{x \in B} \int_\Omega |\hat{x} \circ f| d\mu,$$

where $B$ is the closed unit ball of $X$. If $f$ is bounded, we also define the operator $T_f : X \to L^1(\mu)$ by $T_f(x) = \hat{x} \circ f$ for $x \in X$. It is clear that the operator norm of $T_f$, $\|T_f\|_{op}$, is the same as the Pettis norm of $f$.

If $(f_n)_{n \in \Gamma}$ is a net of weak* measurable functions and if $\Sigma_a$ is a sub $\sigma$-algebra of $\Sigma$ for every $\alpha \in \Gamma$, then we say $(f_n, \Sigma_a)_{\alpha \in \Gamma}$ is a weak* martingale if

(i) $\Sigma_a \subset \Sigma_\beta$ if $\alpha < \beta$,

(ii) for every $x \in X$, $E(\hat{x} \circ f_\beta | \Sigma_\alpha) = \hat{x} \circ f_\alpha$ if $\alpha < \beta$,

where $E(\cdot | \Sigma_\alpha)$ is the usual conditional expectation operator with respect to $\Sigma_\alpha$ (see [3]). This just says that $(\hat{x} \circ f_\alpha, \Sigma_\alpha)_{\alpha \in \Gamma}$ is a scalar-valued martingale for every $x \in X$.

We need the following rather deep result of Fremlin for perfect measure spaces [7]. (See [7, 15] for definitions and properties of perfect spaces.)

**Fremlin's Theorem.** Let $(\Omega, \Sigma, \mu)$ be a finite perfect measure space and $(f_n)_{n=1}^{\infty}$ be a sequence of measurable extended real-valued functions on $\Omega$. Then either $(f_n)_{n=1}^{\infty}$ has a subsequence which converges a.e. or $(f_n)_{n=1}^{\infty}$ has a subsequence having no measurable pointwise cluster points.

We are now able to prove

**Proposition 2.** Let $(\Omega, \Sigma, \mu)$ be a perfect measure space and $X$ a Banach space. Suppose $f : \Omega \to X^*$ is bounded and weak* measurable. Then the following are equivalent:

(i) $T_f : X \to L^1(\mu)$ is a compact operator.

(ii) If $(f_\alpha, \Sigma_\alpha)_{\alpha \in \Gamma}$ is any bounded weak* martingale such that $\hat{x} \circ f_\alpha$ converges to $\hat{x} \circ f$ in $L^1(\mu)$ for every $x \in X$, then $f_\alpha$ converges to $f$ in Pettis norm.

(iii) There exists a net of bounded simple functions $(f_\alpha)_{\alpha \in \Gamma}$, such that $f_\alpha$ converges to $f$ in Pettis norm.

Proof. Without loss of generality $f$ takes its range in $B^*$.

(i) $\Rightarrow$ (ii). Suppose $T_f$ is compact and let $(f_\alpha, \Sigma_\alpha)_{\alpha \in \Gamma}$ be any bounded weak* martingale such that $\hat{x} \circ f_\alpha \to \hat{x} \circ f$ in $L^1(\mu)$. Letting $T_{f_\alpha} : X \to L^1(\mu)$ by $T_{f_\alpha}(x) = \hat{x} \circ f_\alpha$, we note that $T_{f_\alpha}$ converges to $T_f(x)$ in $L^1(\mu)$ and $T_{f_\alpha}(x) = E(\hat{x} \circ f | \Sigma_\alpha)$. It
suffices to show that $T_{f_\alpha}(x)$ converges to $T_f(x)$ uniformly on $B$, as this says

$$\lim_{\alpha} \sup_{x \in B} \|T_{f_\alpha}(x) - T_f(x)\|_{L^1} = 0,$$

or

$$\lim_{\alpha} \sup_{x \in B} \int_{\Omega} |\hat{x} \circ f_\alpha - \hat{x} \circ f| d\mu = 0.$$ 

Let $\varepsilon > 0$. Since $T_f$ is compact, there exists $x_1, \ldots, x_n \in B$ such that $T_f(B) \subseteq \bigcup_{i=1}^n \{ g : \|g - T_f(x_i)\|_{L^1} < \varepsilon/3 \}$. Choose $\beta$ such that if $\alpha > \beta$ then

$$\|T_{f_\alpha}(x_i) - T_f(x_i)\|_{L^1} < \varepsilon/3$$

for $i = 1, \ldots, n$. Let $x \in B$ and let $x_i$ be such that $\|T_f(x) - T_f(x_i)\| < \varepsilon/3$. We note that $E(\cdot | \Sigma_\alpha)$ is an $L^1$ contraction [3]. Then if $\alpha > \beta$,

$$\|T_{f_\alpha}(x) - T_f(x)\|_{L^1} \leq \|T_{f_\alpha}(x) - T_{f_\alpha}(x_i)\|_{L^1} + \|T_{f_\alpha}(x_i) - T_f(x_i)\|_{L^1} + \|T_f(x_i) - T_f(x)\|_{L^1} < \varepsilon.$$

(ii) $\Rightarrow$ (iii). It suffices to show that there exists a bounded weak* martingale $(f_\alpha, \Sigma_\alpha)_{\alpha \in \Gamma}$ such that $f_\alpha$ is simple for every $\alpha \in \Gamma$, and $\hat{x} \circ f_\alpha \to \hat{x} \circ f$ in $L^1$ for every $x \in X$.

Let $\Pi$ be the set of finite partitions of $\Omega$ into elements of $\Sigma$ directed by refinement. If $\pi \in \Pi$, let $\Sigma_\pi$ be the finite $\sigma$-algebra generated by the elements of $\pi$ and let

$$f_\pi = \sum_{A \in \pi} \frac{(w^*)^{-1}}{\mu(A)} f_\pi^{-1} \chi_A.$$ 

It is clear that $(f_\pi, \Sigma_\pi)_{\pi \in \Pi}$ is a weak* martingale, each $f_\pi$ is simple and the fact that $\hat{x} \circ f_\pi \to \hat{x} \circ f$ in $L^1(\mu)$ follows from scalar-valued martingale convergence theorems [3].

(iii) $\Rightarrow$ (i). Suppose $(f_\alpha)_{\alpha \in \Gamma}$ is a net of simple functions converging to $f$ in Pettis norm. Then $T_{f_\alpha}$ converges to $T_f$ in operator norm. Since $T_{f_\alpha}$ is a finite rank operator for each $\alpha$, $T_f$ is compact. $\square$

The following was first observed by Stegall [8].

**Proposition 3.** If $(\Omega, \Sigma, \mu)$ is a perfect finite measure space and $f : \Omega \to X^*$ is bounded and weakly measurable, then $T_f : X \to L^1$ is compact. Hence there exists a net of simple functions converging to $f$ in Pettis norm.

**Proof.** Let $(x_n)^\infty_{n=1}$ be bounded in $X$. Suppose $(\hat{x}_n \circ f)^\infty_{n=1}$ does not have an a.e. convergent subsequence. By Fremlin's theorem, there is a subsequence $(\hat{x}_{n_j} \circ f)^\infty_{j=1}$ having no measurable pointwise cluster points. Let $x^{**}$ be a weak* cluster point of $(\hat{x}_{n_j})^\infty_{j=1}$ in $X^{**}$. Hence $x^{**} \circ f$ is a pointwise cluster point of $(\hat{x}_{n_j} \circ f)^\infty_{j=1}$, and is therefore nonmeasurable. This contradicts the weak measurability of $f$. Hence some subsequence must converge a.e. and by boundedness this subsequence must converge in $L^1(\mu)$. $\square$
3. Main result. Putting together the pieces from §2 yields

**Theorem 4.** If \((\Omega, \Sigma, \mu)\) is a perfect measure space and \(X\) is a Banach space, then \(X^*\) has \(\mu\)-PIP if and only if for every \(f: \Omega \to X^*\) that is bounded and weakly measurable

\[
\left\| (w^*) - \int_{\Omega} f \, d\mu \right\| = \left\| (D) - \int_{\Omega} f \, d\mu \right\|.
\]

In fact if \(f\) is not Pettis integrable, then for some \(E \in \Sigma\) and \(\alpha > 0\) there exists a sequence of simple functions \((f_n)_{n=1}^\infty\) such that

\[
\left\| (w^*) - \int_E f - f_n \, dp \right\| \to 0 \quad \text{but} \quad \left\| (D) - \int_E f - f_n \, d\mu \right\| > \alpha \quad \text{for every } n.
\]

**Proof.** Of course if \(f\) is Pettis integrable then these two norms are the same. Conversely, let \(f: \Omega \to X^*\) be bounded and weakly measurable. Without loss of generality, \(f\) takes its range in \(B^*\). By Lemma 1, it suffices to show that for every \(E \in \Sigma\) and \(\eta > 0\) there exists a finite subset \(A\) of \(X^*\) and \(\epsilon > 0\) such that

\[
\left\| f \right\|_{\infty} < \eta \quad \text{whenever } x^* \notin C_{A, \epsilon}.
\]

Let \(E \in \Sigma\) and \(\eta > 0\). Let \(f_E = f|_E\). Choose by Proposition 3 a simple function \(h\) such that \(\|f_E - h\|_{\infty} < \eta/2\). Note then by hypothesis

\[
\left\| (D) - \int_{\Omega} (f_E - h) \, d\mu \right\| = \left\| (w^*) - \int_{\Omega} f_E - h \, d\mu \right\| \leq \|f_E - h\|_{\infty} < \eta/2.
\]

Hence letting \(A\) be the range of \(h\) and \(\epsilon = \eta/2\), we see that if \(x^* \in C_{A, \epsilon}\), then

\[
\left\| x^* \circ f \right\| \leq \int_{\Omega} x^* \circ (f_E - h) \, d\mu + \int_{\Omega} x^* \circ h \, d\mu
\]

\[
< \left\| (D) - \int_{\Omega} (f_E - h) \, d\mu \right\| + \eta/2 < \eta.
\]

This proves the first assertion. To prove the second, we know that if \(f\) is bounded and weakly measurable, we can always find a sequence of simple functions \((f_n)\) such that \(\| (w^*) - f_n (f - f_n) \|\) converges to zero. If there did not exist an \(\alpha > 0\) such that \(\| (D) - f_n (f - f_n) \| > \alpha\), then this would force \(f\) to be Pettis integrable as in the above argument. \(\square\)

The following example is due to Phillips and is discussed in detail by Geitz in [9 and 10]. Let \((\Omega, \Sigma, \mu)\) be usual Lebesgue measure space and \(l^\infty[0,1]\) be the space of bounded functions with usual supremum norm. Sierpiński constructed a subset \(B\) of \([0,1] \times [0,1]\) such that

(i) for every \(t_0 \in [0,1]\), \(\{ s: (s, t_0) \in B \}\) is countable, and

(ii) for every \(s_0 \in [0,1]\), \(\{ t: (s_0, t) \notin B \}\) is countable.

It is shown in [10] that the function \(f: [0,1] \to l^\infty[0,1]\) given by \([f(s)](t) = \chi_B(s, t)\) is bounded and weakly measurable but not Pettis integrable with respect to \((\Omega, \Sigma, \mu)\).

It is also shown that if \(e_{t_0}\) is the evaluation functional at \(t_0\) on \(l^\infty[0,1]\), then from (i) we have

\[
\int_{[0,1]} e_{t_0} f(s) \, d\mu(s) = \int_{[0,1]} \chi_B(s, t_0) \, d\mu(s) = 0.
\]
Hence, \(\|(w^*)-\int_E f \, d\mu\| = 0\) for every \(E \in \Sigma\). However, if \(\beta \in \text{ba}[0,1] = (l^\infty[0,1])^*\) is such that \(\beta\) vanishes on countable sets and \(\|\beta\| = 1\), then we have by (ii) that, for every \(s_0 \in [0,1]\),

\[
\int_{[0,1]} f(s_0) \, d\beta = 1.
\]

Hence, \(\int_{[0,1]} f(s) \, d\beta \, d\mu(s) = \mu(E)\) for every \(E \in \Sigma\), and thus \(\|(D)-\int_E f \, d\mu\| = \mu(E)\) for every \(E \in \Sigma\).

4. Observations and questions. Suppose \(f : \Omega \to X^*\) is bounded and weak* measurable, that is \(\hat{x} \circ f\) is measurable for every \(x \in X\), such that \(T_f\) is weak compact. Hence, \(T_f^{**} : X^{**} \to L^1(\mu)\). This will certainly, but not necessarily, be the case if \(f\) is weakly measurable. Since \(\langle T_f^{**}(x^{**}), g \rangle = x^{**}((w^*)-\int_{\Omega} f \, dg)\) for every \(g \in L^\infty(\mu)\), there exists a function \(h_{x,**} \in L^1(\mu)\) such that \(x^{**}((w^*)-\int_{\Omega} f \, dg) = \int_{\Omega} h_{x,**} \, dg \, d\mu\) for every \(g \in L^\infty(\mu)\). Note that if \((x_{\beta})\) is a net in \(X\) such that \(\hat{x}_{\beta}\) converges weak* to \(x^{**}\), then clearly \(x^{**} \circ f\) is a pointwise limit of \((\hat{x}_{\beta} \circ f)\), whereas \(h_{x,**}\) is an \(L^1(\mu)\) limit of \((\hat{x}_{\beta} \circ f)\). Consequently, for every \(x^{**}\) there exists a sequence \((x_n)_{n=1}^\infty\) in \(X\) such that \(x_n \circ f\) converges a.e. to \(h_{x,**}\). If \(f\) is Pettis integrable, however, it must be the case that \(x^{**} \circ f = h_{x,**}\) a.e. Hence we get

**Theorem 5.** Let \(X\) be a Banach space and \((\Omega, \Sigma, \mu)\) a finite measure space. Suppose \(f : \Omega \to X^*\) is bounded and weakly measurable. Then \(f\) is Pettis integrable if and only if, for every \(x^{**} \in X^{**}\), there exists a bounded sequence \((x_n)_{n=1}^\infty\) in \(X\) such that both of the following hold:

(i) \(\hat{x}_n \circ f\) converges a.e. to \(x^{**} \circ f\),

(ii) \(\hat{x}_n((w^*)-\int_E f \, d\mu)\) converges to \(x^{**}((w^*)-\int_E f \, d\mu)\) for every \(E \in \Sigma\).

We observe that condition (i) above guarantees that \(\|(w^*)-\int f \, d\mu\| = \|(D)-\int f \, d\mu\|\). Hence we get the following corollary from Theorems 4 and 5:

**Corollary 6.** Let \(X\) be a Banach space and \((\Omega, \Sigma, \mu)\) a perfect finite measure space. Then \(X^*\) has \(\mu\)-PIP if and only if whenever \(f : \Omega \to X^*\) is bounded and weakly measurable then, for every \(x^{**} \in X^{**}\), there is a bounded sequence \((x_n)_{n=1}^\infty\) in \(X\) such that \(\hat{x}_n \circ f\) converges to \(x^{**} \circ f\) a.e.

**Question.** Is it possible to remove condition (ii) in Theorem 5?

**References**


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