PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 2, October 1985

SOME SHARP WEAK-TYPE INEQUALITIES FOR HOLOMORPHIC FUNCTIONS ON THE UNIT BALL OF $\mathbf{C}^n$

BOGUSLAW TOMASZEWSKI

ABSTRACT. Let $B^n = \{z \in \mathbf{C}^n: |z| < 1\}$, $S^n = \partial B^n$ and let σ_n be the Haar measure on S^n. Then for all $f \in H^p$ $(1 \leqslant p < \infty)$ such that $\operatorname{Im}(f(0)) = 0$ and $t > 0$,

$$\sigma_n(\{z \in S^n: |f(z)| \geqslant t\}) \leqslant C_p \cdot \frac{\|\operatorname{Re} f\|_p^p}{t^p}$$

for some constant C_p depending only on p. The best constant C_p is found for $1 \leqslant p \leqslant 2$.

Let $\mathbf{C}^n$ be an n-dimensional complex space with norm

$$\|z\| = \left(|z_1|^2 + \cdots + |z_n|^2\right)^{1/2}$$

and unit ball $B^n = \{z \in \mathbf{C}^n: \|z\| < 1\}$. By σ_n we shall denote the rotation-invariant, normalized Borel measure on $S^n = \partial B^n$. We shall write D and T instead of B^1 and S^1. For a σ_n-measurable function $f: S^n \to \mathbf{C}$ and $p \geqslant 1$, let us define

$$\|f\|_p = \left(\int_{S^n} |f|^p \, d\sigma_n\right)^{1/p}.$$

If $\|f\|_p < \infty$ and if the Poisson integral $P[f]$ of the function f (see [**5**, p. 41]) is a holomorphic function, then we shall write $f \in H^p(S^n)$. Kolmogorov proved [**4**] that there exists a constant $C > 0$ such that, if $f \in H^1(T)$ and if $f(0) = P[f](0)$ is real, then

$$\sigma_1(\{z \in T: |\operatorname{Im} f(z)| \geqslant t\}) \leqslant C \cdot \frac{\|\operatorname{Re} f\|_1}{t}, \tag{1}$$

for all $t > 0$. In other words, the operator $\operatorname{Re} f \to \operatorname{Im} f$ is of the weak type 1-1 (see [**7**]). The best constant of inequality (1) was found by Davis [**3**]. Baernstein [**2**] gave an elementary proof of inequality (1) with the best constant. His proof was modified by Tomaszewski [**6**], who found the best constant in a weak-type inequality for the operator $\operatorname{Re} f \to f$. In this paper we shall prove similar sharp weak-type inequalities for spaces $H^p(S^n)$.

Received by the editors October 23, 1984 and, in revised form, December 28, 1984. The results of this paper were presented during the AMS meeting in San Diego, California, November 9–10, 1984. Presentation was sponsored by the Oklahoma State University.

1980 *Mathematics Subject Classification*. Primary 32-XX, 30-XX.

THEOREM. *If* $1 \leqslant p \leqslant 2, f \in H^p(S^n)$ *and* $\operatorname{Im} f(0) = 0$, *then for all* $t > 0$,

$$\sigma_n\big(\{z \in S^n: |f(z)| \geqslant t\}\big) \leqslant C_p \cdot \frac{\|\operatorname{Re} f\|_p^p}{t^p}, \tag{2}$$

where

$$C_p = \frac{\sqrt{\pi}}{2} \cdot \frac{p \cdot \Gamma(p/2)}{\Gamma((p+1)/2)}.$$

The constant C_p *is the best possible in this inequality.*

We shall need the following

LEMMA. *Let* u_p, *for* $1 \leqslant p \leqslant 2$, *be the Poisson integral of the function* $\gamma_p(e^{it}) = |\cos t|^p$ *defined on* T. *The inequalities*

(i) $u_p(z) \leqslant u_p(0) + |\operatorname{Re} z|^p$,

(ii) $u_p(0) \leqslant u_p(x)$

hold for $z \in D$ *and* $-1 \leqslant x \leqslant 1$.

PROOF. Let $u_p(z) = \operatorname{Re}(\sum_{k=0}^{\infty} a_k \cdot z^k)$ for some real numbers a_k. It is easy to see that $a_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$. We shall prove that

$$(-1)^n \cdot a_{2n} \leqslant 0, \tag{3}$$

for $n = 1, 2, \ldots$. We have

$$\begin{aligned}(-1)^n \cdot a_{2n} &= \frac{(-1)^n}{\pi} \cdot \int_{-\pi}^{\pi} |\cos t|^p \cdot \cos 2nt \, dt \\ &= \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} |\sin t|^p \cdot \cos 2nt \, dt \\ &= \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} (1 - \cos^2 t)^{p/2} \cdot \cos 2nt \, dt \\ &= \frac{1}{\pi} \cdot \sum_{m=1}^{\infty} b_m \cdot \int_{-\pi}^{\pi} \cos^{2m} t \cdot \cos 2nt \, dt = \frac{1}{\pi} \cdot \sum_{m=1}^{\infty} b_m \cdot I_{2m,2n},\end{aligned}$$

where $b_m < 0$ are real numbers such that $(1 - s)^{p/2} = 1 + \sum_{m=1}^{\infty} b_m \cdot s^m$ and $I_{m,n} = \int_{-\pi}^{\pi} \cos^n t \cdot \cos mt \, dt$. Since $I_{m,n} = \frac{1}{2} \cdot (I_{m-1,n-1} + I_{m-1,n+1})$ it can be easily proved (by induction on m) that $I_{m,n} \geqslant 0$. This ends the proof of inequality (3). Now, let us note that for $z = x + iy \in D$,

$$\begin{aligned}&u_p(z) - u_p(0) - a_2 \cdot (x^2 - y^2) \\ &= \operatorname{Re}\left(\sum_{m=2}^{\infty} a_{2m} \cdot z^{2m}\right) \leqslant (x^2 + y^2) \cdot \sum_{m=2}^{\infty} |a_{2m}| \\ &= (x^2 + y^2) \cdot \big(u_p(0) - u_p(i) - a_2\big) = (x^2 + y^2) \cdot \frac{2-p}{2p} \cdot a_2,\end{aligned}$$

since

$$a_0 = \frac{2}{\pi} \cdot \frac{\Gamma((p+1)/2) \cdot \Gamma(1/2)}{p \cdot \Gamma(p/2)},$$

$$a_2 = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} |\cos t|^{p+2}\, dt - 2a_0 = \frac{2p}{p+2} \cdot a_0.$$

But $(2-p)/2p \leqslant 1$ and

$$x^2 \cdot ((2-p)/2p) \cdot a_2 + x^2 \cdot a_2 = a_0 \cdot x^2 \leqslant x^2 \leqslant x^p.$$

Hence, inequality (i) follows. Let us turn to (ii). For $0 \leqslant \alpha \leqslant \pi/2$ let v_α be the Poisson integral of the characteristic function, defined on T, of the set $\{z \in T\colon |\mathrm{Im}\, z| \geqslant \sin\alpha\}$. Then

$$v_\alpha(z) = \frac{1}{\pi} \cdot \mathrm{Arg}\left(\frac{1 - ia - z - ia}{1 - ia + z + ia}\right) + \frac{1}{\pi} \cdot \mathrm{Arg}\left(\frac{1 + iaz + z - ia}{1 + iaz - z + ia}\right) + 1,$$

where $a = (\cos\alpha)/(1 + \sin\alpha)$. We see that

$$v_\alpha(x) = \frac{2}{\pi} \cdot \mathrm{Arg}\left[1 - a^2 - ia\left(\frac{1+x}{1-x} + \frac{1-x}{1+x}\right)\right] + 1 \leqslant v_\alpha(0).$$

But $u_p(z) = 1 - \int_0^{\pi/2} v_\alpha(z)\, d\mu(\alpha)$ for some positive measure μ ($d\mu(t) = p \cdot \cos^{p-1} t \cdot \sin t\, dt$). Thus, inequality (ii) follows.

Proof of the Theorem. We shall first prove that equality holds in (2) for inner functions $f\colon S^n \to \mathbf{C}$ such that $f(0) = 0$ (i.e. functions $f \in H^p(S^n)$ such that $|f(z)| = 1$ a.e. on S^n and $\int_{S^n} f(z)\, d\sigma_n(z) = 0$). The existence of such functions was proved by Aleksandrov [1]. If f is inner and $f(0) = 0$, then

$$\int_{S^n} h(f(z))\, d\sigma_n(z) = \int_T h(z)\, d\sigma_1(z),$$

for every continuous function h defined on T (see [5, p. 405]). Taking $h(z) = |\mathrm{Re}\, z|^p$, we see (2) cannot hold with any constant smaller than the constant C_p defined above.

We shall prove inequality (2) for the case $n = 1$. For each function $f \in H^p(S^n)$, we have

$$\|\mathrm{Re}\, f\|_p^p = \int_{S^n} \int_T |\mathrm{Re}\, f(\xi z)|^p\, d\sigma_1(\xi)\, d\sigma_n(z).$$

If we now apply the statement of the Theorem, for the case $n = 1$, to the integrals $\int_T |\mathrm{Re}\, f(\xi z)|^p\, d\sigma_1(\xi)$, we shall get the general case. Thus, let us assume that $f \in H^p(T)$. Let $\varphi\colon \mathbf{C} \to R$ be a function such that $\varphi(z) = |\mathrm{Re}\, z|^p$ for $z \notin D$ and $\varphi(z) = u_p(z)$ for $z \in D$, where u_p is defined in the lemma. The function φ is continuous and since the function $|\mathrm{Re}\, z|^p$ is subharmonic, we have $\varphi(z) \geqslant |\mathrm{Re}\, z|^p$ for $z \in \mathbf{C}$. It follows that φ is subharmonic on $\mathbf{C}$. Let $E = \{z \in T\colon |f(z)| \geqslant 1\}$ and let us define the functions $\omega(z) = \mathrm{P}[\chi_{T\setminus E}](z)$, $h(z) = \mathrm{P}[|\mathrm{Re}\, f|^p](z)$, where $\chi_{T\setminus E}$ is a characteristic function of the set E and P denotes the Poisson integral for the unit disk D. We shall prove that

(4) $$\varphi(f(z)) \leqslant h(z) + \varphi(0) \cdot \omega(z),$$

where we write $f(z) = \mathrm{P}[f](z)$. It suffices to check this inequality for $z \in T$, since the function $\varphi \circ f$ is subharmonic and the function $h(z) + \varphi(0) \cdot \omega(z)$ is harmonic. If $z \in T - E$, then $f(z) \in D$. Hence, for this case (4) follows from (i) and the definition of the function h and the function ω. If $z \in E$, then $\varphi(f(z)) = |\mathrm{Re}\, f(z)|^p = h(z)$, and (4) also holds for this case, hence for every $z \in T$. Taking $z = 0$ in (4) and applying (ii), we get

$$\begin{aligned}\varphi(0) \leqslant \varphi(f(0)) &\leqslant h(0) + \varphi(0) \cdot \omega(0) \\ &= \|\mathrm{Re}\, F\|_p + \varphi(0) \cdot \sigma_1(T - E).\end{aligned}$$

This ends the proof of inequality (2) for the case $t = 1$, since $\sigma_1(T - E) = 1 - \sigma_1(E)$ and $\varphi(0) = u_p(0) = (C_p)^{-1}$. A general case can be proved by considering the function f/t instead of f.

References

1. A. B. Aleksandrov, *The existence of inner functions in the ball*, Mat. Sb. (N.S.) **118** (1982), 147–163 (Russian); Math. USSR-Sb. **43** (1983), 143–159.

2. A. Baernstein, *Some sharp inequalities for conjugate functions*, Indiana Univ. Math. J. **27** (1978), 833–852.

3. B. Davis, *On the weak-type* (1-1) *inequality for conjugate functions*, Proc. Amer. Math. Soc. **44** (1974), 307–311.

4. A. N. Kolmogorov, *Sur les fonctions harmoniques conjugées et la séries de Fourier*, Fund. Math. **7** (1925), 23–28.

5. W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer-Verlag, New York, 1980.

6. B. Tomaszewski, *The best constant in a weak-type* H^1*-inequality, complex analysis and its applications*, Complex Variables, Vol. 4, 1984, pp. 35–38.

7. A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, London and New York, 1968.

Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74078

Current address: Department of Mathematics, State University of New York, 1400 Washington Avenue, Albany, New York 12222