WEIGHTED REVERSE WEAK TYPE INEQUALITIES
FOR THE ERGODIC MAXIMAL FUNCTION
AND THE CLASSES $L \log^+ L$

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Abstract. D. Ornstein proved that the ergodic maximal function satisfies a reverse weak type inequality, and from this he deduced that the integrability of the maximal function $f^*$ implies that $f$ belongs to $L \log^+ L$. Weighted analogues of these results are proved.

Let $(\Omega, \Sigma, \mu)$ denote a probability space and let $T: \Omega \to \Omega$ be an invertible, measure preserving ergodic transformation. The Maximal Ergodic Theorem asserts that the maximal function $f^*$ defined for nonnegative $f \in L^1(\mu)$ by

$$f^*(x) = \sup_{m,n \geq 0} \frac{1}{m+n+1} \sum_{k=-m}^{n} f(T^k x), \quad x \in \Omega,$$

satisfies the weak type inequality

$$\mu \{ x : f^*(x) > \lambda \} \leq \lambda^{-1} \int_{\{ x : f^*(x) > \lambda \}} f(x) \, d\mu(x) \leq \lambda^{-1} \int \Omega f(x) \, d\mu(x)$$

for all $\lambda > 0$. On the other hand, D. Ornstein [6] has shown that $f^*$ also satisfies the reverse weak type inequality

$$\mu \{ x : f^*(x) > \lambda \} \geq (2\lambda)^{-1} \int_{\{ x : f^*(x) > \lambda \}} f(x) \, d\mu(x)$$

for all $\lambda$ such that $\mu \{ x : f^*(x) > \lambda \} < 1$; in particular, this is the case if $\lambda > \lambda_f = \int_{\Omega} f(x) \, d\mu(x)$. From this he deduced that if $f^*$ is integrable then $f$ belongs to the class $L \log^+ L$. For elementary proofs of these results see the recent papers of R. L. Jones [3 and 4]. The purpose of this paper is to prove the following weighted version of Ornstein's result.

Theorem. Suppose $u$ and $v$ are nonnegative measurable functions on $\Omega$ with $u \in L^1(\mu)$. The following statements are equivalent:

(i) There is a constant $C$ independent of $f$ such that

$$(1) \quad \int_{\{ x : f^*(x) > \lambda \}} u(x) \, d\mu(x) \geq (C\lambda)^{-1} \int_{\{ x : f^*(x) > \lambda \}} f(x) v(x) \, d\mu(x)$$

holds for all $\lambda$ such that $\mu \{ x : f^*(x) > \lambda \} < 1$. 

Received by the editors December 18, 1984.

1980 Mathematics Subject Classification. Primary 28D05; Secondary 42B25.

Key words and phrases. Maximal functions, ergodic maximal function, weighted inequalities, $L \log^+ L$.

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0002-9939/85 $1.00 + $.25 per page

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(ii) There is a constant $D$ such that

$$v(x) \leq D \frac{1}{2n+1} \sum_{k=-n}^{n} u(T^k x)$$

holds for all $n \geq 0$ and almost all $x \in \Omega$.

**Corollary.** Suppose $u$ and $v$ satisfy (2) and that $\int_{\Omega} f^*(x) u(x) \, d\mu(x) < \infty$. Then $f$ satisfies

$$\int_{\Omega} \left[ f(x) \log^+ f(x) \right] v(x) \, d\mu(x) < \infty.$$ 

An analogue of the Theorem for the Hardy-Littlewood maximal function in $\mathbb{R}^n$ was recently obtained by B. Muckenhoupt [5].

Examples of weight function pairs which satisfy (2) may be constructed as follows. A nonnegative weight function $w$ is said to satisfy the $A_1$ condition [2, 7] if there is a constant $C$ such that $w^*(x) \leq C w(x)$ a.e. in $\Omega$. If $g \in L^1$ and $0 < \delta < 1$, it is shown in [7] that $w(x) = g^*(x)^\delta$ belongs to $A_1$. Now if $w$ belongs to $A_1$, the Schwarz inequality for sums shows that

$$\left( 2n + 1 \right)^2 \leq \left( \sum_{k=-n}^{n} w(T^k x) \right) \left( \sum_{k=-n}^{n} 1/w(T^k x) \right) \leq (2n+1) w^*(x) \left( \sum_{k=-n}^{n} 1/w(T^k x) \right) \leq C(2n+1) w(x) \left( \sum_{k=-n}^{n} 1/w(T^k x) \right).$$

Thus (2) is satisfied if $u(x) = v(x) = 1/w(x)$.

Let $Z$ denote the set of integers and, for any finite subset $I$ of $Z$, let $|I|$ denote its cardinality. If $I$ consists of finitely many consecutive integers we say that $I$ is an interval. If $I$ is an interval, we write $2I$ for the largest interval containing $I$ in its center and satisfying $|2I| < 2|I|$. Thus $2I$ is obtained by adjoining to $I$ two intervals (possibly empty) of equal cardinality, one of which immediately precedes and one of which immediately succeeds $I$. As usual, if $E$ is any set, $\chi_E(x)$ denotes the characteristic function of $E$.

We need the following lemma.

**Lemma.** Let $I$ be an interval in $Z$ and let $J$ denote the complement of $I$ in $Z$. Then $I$ is the union of $N$ pairwise disjoint intervals $I_j$, which satisfy

(i) $|I_j|/2 \leq \text{dist}(I_j, J) \leq 4|I_j|$, $1 \leq j \leq N$,

(ii) $\sum_{j} \chi_{2I_j}(n) \leq 4\chi_I(n)$ for all $n \in Z$.

**Proof of the Lemma.** Suppose $|I| > 2$. Otherwise the Lemma is trivially satisfied by choosing intervals $I_j$, each consisting of one integer. Let $I_1$ be the largest interval centered in $I$ with $|I_1| \leq |I|/2$. Then $I \setminus I_1$ consists of two intervals, say $I'$ and $I''$ with $|I'| = |I''| \geq |I|/4$. We shall show that $I'$ is a union of pairwise disjoint intervals $I_j$, $j = 2, \ldots, N'$, such that (i) holds. If $|I'| \leq 2$ we may write $I'$ as a union
of intervals each consisting of one integer and our selection procedure terminates; otherwise \( I' \) is the union of two intervals \( I'_- \) and \( I'_+ \) with \( d(I'_-, J) = 1 \) and \( |I'_+| - 1 \leq |I'_-| \leq |I'_+| \). Then \( I'_- \) is the union of two intervals \( I_2 \) and \( I_3 \) satisfying \( |I_2| - 1 \leq |I_3| \leq |I_2| \) and \( \text{dist}(I_3, I') = 1 \). Clearly, (i) is satisfied for \( j = 2, 3 \). This selection procedure is now repeated with \( I'_+ \) in place of \( I' \) and, continuing in this way, \( I' \) is eventually exhausted by the intervals \( I_j, j = 2, \ldots, N' \). It is easy to see that any element of \( I \) belongs to at most 3 of the intervals \( 2I_j, 2 \leq j \leq N' \), and that all such \( 2I_j \) are disjoint from \( I'' \). A similar construction is applied to express \( I'' \) as a union of intervals \( I_j, j = N' + 1, \ldots, N \). The intervals \( I_j, j = 1, \ldots, N \), so constructed clearly satisfy (ii).

**Proof of the Theorem.** Suppose that (1) holds. We will show that (2) holds with \( D = 2C \). Let \( A \in \Sigma \) with \( 0 < \mu(A) < 1/2 \). Then \( (\chi_A)^*(x) < 1 \) if \( x \notin A \), so the theorem of dominated convergence together with (1) shows

\[
\int_A u(x) \, d\mu(x) = \lim_{\lambda \to 1^-} \int_{\{x: (\chi_A)^*(x) > \lambda\}} u(x) \, d\mu(x) \
\geq \lim_{\lambda \to 1^-} \frac{1}{\lambda} \int_{\{x: (\chi_A)^*(x) > \lambda\}} \chi_A(x) v(x) \, d\mu(x) \
= \frac{1}{C} \int_A v(x) \, d\mu(x).
\]

Since \( A \) is arbitrary, this shows that \( u(x) \geq v(x)/C \) a.e. Thus (2) holds with \( D = 2C \) for the case \( n = 0 \). Now let \( n \geq 1 \). If \( A \in \Sigma \) with \( \mu(A) > 0 \) there is a subset \( B \) of \( A \) with \( \mu(B) > 0 \) such that \( T^j(B) \), \( j = -n, \ldots, n \), are pairwise disjoint. If \( x \notin \bigcup_{j=-n}^n T^j(B) \), then any 1 which occurs in the sequence \( \chi_B(T^jx) \) is both followed and preceded by at least \( n \) consecutive zeros; furthermore, \( \chi_B(T^kx) = 0 \) for \( -n \leq k \leq n \). For such \( x \) it then follows that \( (\chi_B)^*(x) \leq 1/(n+1) \) and hence \( \{x: (\chi_B)^*(x) > 1/(n + 1)\} \) is a subset of \( \bigcup_{j=-n}^n T^j(B) \). Thus, (1) implies

\[
\int_B \sum_{j=-n}^n u(T^jx) \, d\mu(x) = \int_{\bigcup_{j=-n}^n T^j(B)} u(x) \, d\mu(x) \
\geq \int_{\{x: (\chi_B)^*(x) > 1/(n+1)\}} u(x) \, d\mu(x) \
\geq \frac{n + 1}{C} \int_{\{x: (\chi_B)^*(x) > 1/(n+1)\}} \chi_B(x) v(x) \, d\mu(x) \
= \frac{n + 1}{C} \int_B v(x) \, d\mu(x).
\]

Since \( A \) was arbitrary, it follows that (2) holds with \( D = 2C \) for \( n \geq 1 \).

Suppose now that (2) holds. We will show that (1) holds with \( C = 20D \).

Observe first that if \( I \) is any interval, then (2) implies that

\[
\sum_{j \in I} u(T^jx) \geq D^{-1} |I| v(T^kx) \quad \text{a.e.}
\]

for all \( k \in I \).
Let $\lambda > \lambda_f$ and $E = \{x: f^*(x) > \lambda\}$. Since $T$ is ergodic it follows that for almost all $x \in E$ there are positive integers $r = r(x)$ and $s = s(x)$ such that $T^j x \in E$ if $-r + 1 \leq j \leq s - 1$ but $T^j x \notin E$ for $j = -r$ and $j = s$. For each positive integer $i$ let $R_i = \bigcup_{j=0}^{i-1} T^j(B_i)$ where $B_i = \{x \in E: r(x) = 1 \text{ and } s(x) = i\}$. Then $\{R_i\}$ is a sequence of pairwise disjoint subsets of $E$, and for almost all $x \in E$ there is $i$, namely $i = r(x) + s(x) - 1$, such that $x \in R_i$. Thus, $E = \bigcup_{i=1}^{\infty} R_i$ and

\[
\int_E f(x) v(x) \, d\mu(x) = \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \int_{B_i} f(T^j x) v(T^j x) \, d\mu(x).
\]

Let $i$ be fixed and let $I_j$ denote the intervals generated by the Lemma for the interval $I = \{0, \ldots, i - 1\}$. Then (3) shows that

\[
\sum_{j=0}^{i-1} f(T^j x) v(T^j x) = \sum_{j,k \in I_j} f(T^j x) v(T^j x) \leq \sum_{j,k \in I_j} f(T^j x) \left( D|I_j|^{-1} \sum_{m=2|I_j|} u(T^m x) \right).
\]

Now part (i) of the Lemma shows that for each $j$ there is an interval $K_j$ with $|K_j| < 5|I_j|$ containing $I_j$ and either $-1$ or $i$. Since neither $T^{-1} x$ nor $T^i x$ belongs to $E$, it follows that

\[
\sum_{k \in I_j} f(T^k x) \leq \sum_{k \in K_j} f(T^k x) \leq |K_j| \lambda < 5|I_j| \lambda.
\]

Using this on the right side of (5) shows, together with (ii) of the Lemma, that

\[
\sum_{j=0}^{i-1} f(T^j x) v(T^j x) \leq 5D\lambda \sum_{j,m \in 2|I_j|} u(T^m x) \leq 20D\lambda \sum_{m=0}^{i-1} u(T^m x)
\]

and, hence,

\[
\int_{R_i} f(x) v(x) \, d\mu(x) = \sum_{j=0}^{i-1} \int_{B_i} f(T^j x) v(T^j x) \, d\mu(x) \leq 20D\lambda \sum_{m=0}^{i-1} \int_{R_i} u(T^m x) \, d\mu(x) = 20D\lambda \int_{R_i} u(x) \, d\mu(x).
\]

In view of (4), summing this over the index $i$ yields (1) with $C = 20D$.

**Proof of the Corollary.** Observe first that (2) implies $v(x) \leq Du(x)$ a.e. and thus we have

\[
\int_\Omega f(x) v(x) \, d\mu(x) \leq \int_\Omega f^*(x) v(x) \, d\mu(x) \leq D \int_\Omega f^*(x) u(x) \, d\mu(x) < \infty.
\]
Now if $\lambda > \lambda_f$, then $\mu \{ x : f^*(x) > \lambda \} < 1$, so (1) shows that

$$\int_\Omega \left[ f(x) \log^+ \left( \frac{f(x)}{\lambda_f} \right) \right] v(x) \, d\mu(x) = \int_{\lambda_f}^{\infty} \frac{d\lambda}{\lambda} \int_{\{ x : f(x) > \lambda \}} f(x) v(x) \, d\mu(x)$$

is bounded above by

$$C \int_{\lambda_f}^{\infty} d\lambda \int_{\{ x : f^*(x) > \lambda \}} u(x) \, d\mu(x) \leq C \int_\Omega f^*(x) u(x) \, d\mu(x) < \infty$$

and this, together with (6), shows that

$$\int_\Omega \left[ f(x) \log^+ f(x) \right] v(x) \, d\mu(x) < \infty.$$

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