SLLN AND CONVERGENCE RATES
FOR NEARLY ORTHOGONAL SEQUENCES
OF RANDOM VARIABLES

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Abstract. Let \( \{X_k : k \geq 1\} \) be a sequence of random variables with finite second moments \( EX_k^2 = \sigma_k^2 < \infty \) for which \( |EX_k X_l| \leq \sigma_l \sigma_k \rho(|k - l|) \), where \( \{\rho(j) : j \geq 0\} \) is a sequence of nonnegative numbers with \( \sum_{j=0}^{\infty} \rho(j) < \infty \). In particular, in the case of orthogonality, \( \rho(j) = 0 \) for \( j \geq 1 \). We prove strong laws for the first arithmetic means \( \tau_n = n^{-1} \sum_{k=1}^{n} X_k \) and the Cesàro means

\[
\tau_n = n^{-1} \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right) X_k.
\]

The convergence rates are studied in the form \( P\{\sup_{n \geq 2p} |\tau_n| > \epsilon\} \) and \( P\{\sup_{n \geq 2p} |\tau_n| > \epsilon\} \), where \( \epsilon > 0 \) is fixed and \( p \) tends to \( \infty \). At the end, the case where \( \sum_{j=0}^{\infty} \rho(j) = \infty \) is also treated.

1. Introduction. Let \( \{X_k : k \geq 1\} \) be an orthogonal sequence of random variables (rv's), i.e.

\[
EX_k X_l = 0 \quad (k \neq l; k, l \geq 1)
\]

with finite second moments

\[
EX_k^2 = \sigma_k^2 \quad (k \geq 1).
\]

We will consider the first arithmetic means \( \xi_n = (1/n) \sum_{k=1}^{n} X_k \) as well as the Cesàro means (of order 1)

\[
\tau_n = \frac{1}{n} \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right) X_k \quad (n \geq 1).
\]

A consequence of the Rademacher-Menshov theorem, well known in the theory of orthogonal series, is the following (see e.g. [3, pp. 86, 87]).

**Theorem A.** If \( \{X_k\} \) is an orthogonal sequence of rv's with (1.2) and

\[
\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} \left[\log(k + 1)\right]^2 < \infty,
\]

then

\[
\lim_{n \to \infty} \xi_n = 0 \quad a.s.
\]
In this paper the logarithms are to the base 2.

It is also pointed out that the sufficient condition (1.3) is the best possible (see Tandori [4]).

The next theorem is due to the author [2].

**THEOREM B.** If \( \{X_k\} \) is an orthogonal sequence of rv’s with (1.2) and

\[
\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} < \infty,
\]

then

\[
\lim_{n \to \infty} \tau_n = 0 \quad a.s.
\]

**2. Main results.** The orthogonality condition (1.1) can be weakened in Theorems A and B maintaining conclusions (1.4) and (1.6), respectively.

To be more precise, we say that the sequence \( \{X_k\} \) of rv’s satisfying condition (1.2) is quasi-orthogonal (or nearly orthogonal) if there exists a sequence \( \{\rho(j): j \geq 0\} \) of nonnegative numbers such that

\[
|E X_k X_l| \leq \sigma_k \sigma_l \rho(|k - l|) \quad (k, l \geq 1)
\]

and

\[
\sum_{j=0}^{\infty} \rho(j) < \infty.
\]

If \( EX_k = 0 \) \( (k \geq 1) \), then (2.1) is equivalent to

\[
|\text{Corr}(X_k, X_l)| \leq \rho(|k - l|) \quad (k, l \geq 1).
\]

Also, we may assume that \( \rho(0) = 1 \) and \( 0 \leq \rho(j) \leq 1 \) \( (j \geq 1) \).

Now, the generalizations of Theorems A and B are the following:

**THEOREM 1.** If \( \{X_k\} \) is a quasi-orthogonal sequence of rv’s, then (1.3) implies (1.4).

**THEOREM 2.** If \( \{X_k\} \) is a quasi-orthogonal sequence of rv’s, then (1.5) implies (1.6).

Both theorems will be obtained as corollaries of the next two theorems stating convergence rates.

**THEOREM 3.** If \( \{X_k\} \) is a quasi-orthogonal sequence of rv’s, then (1.3) implies, for every \( \epsilon > 0 \),

\[
P \left( \sup_{n \geq 2^p} |s_n| > \epsilon \right) = O \left( \frac{1}{2^p} \sum_{k=1}^{2^p} \sigma_k^2 + \sum_{k=2^p+1}^{\infty} \frac{\sigma_k^2}{k^2} \left[ \log(k+1) \right]^2 \right) \quad (p \geq 0).
\]

**THEOREM 4.** If \( \{X_k\} \) is a quasi-orthogonal sequence of rv’s, then (1.5) implies, for every \( \epsilon > 0 \),

\[
P \left( \sup_{n \geq 2^p} |\tau_n| > \epsilon \right) = O \left( \frac{1}{2^p} \sum_{k=1}^{2^p} \sigma_k^2 + \sum_{k=2^p+1}^{\infty} \frac{\sigma_k^2}{k^2} \right) \quad (p \geq 0).
\]

We note two other consequences of Theorems 3 and 4.
Corollary 1. Assume \{X_k\} is a quasi-orthogonal sequence of rv's and

\begin{equation}
\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} [\log(k+1)]^2 \lambda(k) < \infty,
\end{equation}

where \{\lambda(k): k \geq 1\} is a nondecreasing sequence of positive numbers such that the sequence \{k^2/\lambda(k): k \geq 1\} is also nondecreasing and tends to \infty. Then, for every \(\varepsilon > 0\),

\begin{equation}
\lim_{p \to \infty} \lambda(2^p) P\left( \sup_{n > 2^p} |\tau_n| > \varepsilon \right) = 0.
\end{equation}

Corollary 2. Assume \{X_k\} is a quasi-orthogonal sequence of rv's and

\begin{equation}
\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} \lambda(k) < \infty,
\end{equation}

where \{\lambda(k)\} is the same as in Corollary 1. Then, for every \(\varepsilon > 0\),

\begin{equation}
\lim_{p \to \infty} \lambda(2^p) P\left( \sup_{n > 2^p} \tau_n > \varepsilon \right) = 0.
\end{equation}

We briefly indicate, e.g., how Corollary 1 can be deduced from Theorem 3.

It follows from (2.5) that (even dropping the factor \([\log(k+1)]^2\), via the Kronecker lemma (see e.g. [3, p. 35]),

\begin{equation}
\lim_{p \to \infty} \frac{\lambda(2^p)}{2^{2p}} \sum_{k=1}^{2^p} \sigma_k^2 = 0.
\end{equation}

On the other hand, again by (2.5),

\begin{equation}
\lim_{p \to \infty} \lambda(2^p) \sum_{k=2^p+1}^{\infty} \frac{\sigma_k^2}{k^2} [\log(k+1)]^2
\leq \lim_{p \to \infty} \sum_{k=2^p+1}^{\infty} \frac{\sigma_k^2}{k^2} [\log(k+1)]^2 \lambda(k) = 0.
\end{equation}

That is, (2.3) implies (2.6).

3. Auxiliary results.

Lemma 1. If \{X_k\} is a sequence of rv's satisfying conditions (1.2) and (2.1), then

\begin{equation}
E \left[ \sum_{k=a+1}^{a+n} X_k \right]^2 \leq \left( 1 + 2 \sum_{j=1}^{n-1} \rho(j) \right) \sum_{k=a+1}^{a+n} \sigma_k^2 \quad (a \geq 0, n \geq 1).
\end{equation}
PROOF. Squaring out, using (1.2) and (2.1), letting \( j = l - k \) and, finally, applying the Cauchy inequality yields (3.1):

\[
E \left[ \sum_{k=\alpha+1}^{a+n} X_k \right]^2 = \sum_{k=\alpha+1}^{a+n} EX_k^2 + 2 \sum_{\alpha+1}^{a+n-1} \sum_{l=k+1}^{a+n} EX_k X_l \\
\leq \sum_{k=\alpha+1}^{a+n} \sigma_k^2 + 2 \sum_{k=\alpha+1}^{a+n-1} \sum_{l=k+1}^{a+n} \sigma_k \sigma_l \rho(l - k) \\
= \sum_{k=\alpha+1}^{a+n} \sigma_k^2 + 2 \sum_{j=1}^{n-1} \rho(j) \sum_{k=\alpha+1}^{a+n-j} \sigma_k \sigma_{k+j} \\
\leq \left( 1 + 2 \sum_{j=1}^{n-1} \rho(j) \right) \sum_{k=\alpha+1}^{a+n} \sigma_k^2,
\]

since

\[
\sum_{k=\alpha+1}^{a+n-j} \sigma_k \sigma_{k+j} \leq \left( \sum_{k=\alpha+1}^{a+n-j} \sigma_k^2 \sum_{k=\alpha+1}^{a+n-j} \sigma_{k+j}^2 \right)^{1/2} \\
= \left( \sum_{k=\alpha+1}^{a+n-j} \sigma_k^2 \sum_{k=\alpha+j+1}^{a+n} \sigma_k^2 \right)^{1/2} \leq \sum_{k=\alpha+1}^{a+n} \sigma_k^2.
\]

In the proof of Theorem 4 we need a slightly more general form of Lemma 1.

**Lemma 2.** If \( \{X_k\} \) is a sequence of rv's satisfying conditions (1.2) and (2.1), and \( \{b_k: k \geq 1\} \) is a sequence of numbers, then

\[
E \left[ \sum_{k=\alpha+1}^{a+n} b_k X_k \right]^2 \leq \left( 1 + 2 \sum_{j=1}^{n-1} \rho(j) \right) \sum_{k=\alpha+1}^{a+n} b_k^2 \sigma_k^2 \quad (a > 0, n \geq 1).
\]

Indeed, applying Lemma 1 for \( \{Y_k = b_k X_k\} \), we get immediately Lemma 2.

The next lemma is a special case of the maximal inequality in [1, Theorem 3].

**Lemma 3.** If \( \{X_k\} \) is a sequence of rv's such that condition (3.1) is satisfied, then

\[
E \left[ \max_{1 \leq m \leq n} \left| \sum_{k=\alpha+1}^{a+m} X_k \right| \right]^2 \leq \left[ \log 2n \right]^2 \left( 1 + 2 \sum_{j=1}^{n-1} \rho(j) \right) \sum_{k=\alpha+1}^{a+n} \sigma_k^2 \\
\quad (a \geq 0, n \geq 1).
\]

**4. Proofs of Theorems 3 and 4.**

**Proof of Theorem 3.** Obviously,

\[
P\left( \sup_{n > 2^p} |\xi_n| > \varepsilon \right) \leq \sum_{q=p}^\infty P\left( \max_{2^q < n \leq 2^{q+1}} |\xi_n| > \varepsilon \right).
\]

A simple estimate shows

\[
\max_{2^q < n \leq 2^{q+1}} |\xi_n| \leq |\xi_{2^q+1}| + \frac{1}{2^q} \max_{2^q < n \leq 2^{q+1}} \left| \sum_{k=2^q+1}^{n} X_k \right|.
\]
On one hand, by (3.1) and (2.2),

\[
(4.2) \quad E\xi_{2^q}^2 = \frac{O(1)}{2^{2q}} \sum_{k=1}^{2^q} \sigma_k^2.
\]

On the other hand, by (3.3) and (2.2),

\[
E \left[ \max_{2^q < n \leq 2^{q+1}} \left| \sum_{k=2^q+1}^{n} X_k \right| \right]^2 = O(1) \left[ \log 2^{q+1} \right]^2 \sum_{k=2^q+1}^{2^{q+1}} \sigma_k^2
\]

\[
= O(1) \sum_{k=2^q+1}^{2^{q+1}} \sigma_k^2 \left[ \log (k + 1) \right]^2.
\]

Thus, by the Chebyshev inequality,

\[
(4.3) \quad P\left\{ \max_{2^q < n \leq 2^{q+1}} |\xi_n| > \varepsilon \right\} \leq P\left\{ |\xi_{2^q}| > \frac{\varepsilon}{2} \right\} + P\left\{ \max_{2^q < n \leq 2^{q+1}} \left| \sum_{k=2^q+1}^{n} X_k \right| > \varepsilon 2^{q-1} \right\}
\]

\[
= O\left( \frac{1}{\varepsilon^2} \right) \left( \frac{1}{2^{2q}} \sum_{k=1}^{2^q} \sigma_k^2 + \frac{1}{2^{2q}} \sum_{k=2^q+1}^{2^{q+1}} \sigma_k^2 \left[ \log (k + 1) \right]^2 \right).
\]

Keeping (4.1) in mind, simple calculations show

\[
(4.4) \quad \sum_{q=p}^{\infty} \frac{1}{2^{2q}} \sum_{k=1}^{2^q} \sigma_k^2 = \sum_{q=p}^{\infty} \frac{1}{2^{2q}} \sum_{k=2^q+1}^{2^{q+1}} \sigma_k^2 \leq \frac{4}{3} \frac{1}{2^{2p}} \sum_{k=2^p+1}^{2^{p+1}} \sigma_k^2
\]

and

\[
(4.5) \quad \sum_{q=p}^{\infty} \frac{1}{2^{2q}} \sum_{k=2^q+1}^{2^{q+1}} \sigma_k^2 \left[ \log (k + 1) \right]^2 \leq \frac{4}{3} \sum_{k=2^p+1}^{\infty} \frac{\sigma_k^2}{k^2} \left[ \log (k + 1) \right]^2.
\]

Collecting (4.1) and (4.3)–(4.5) yields (2.3).

PROOF OF THEOREM 4. Similarly to (4.1),

\[
(4.6) \quad P\left\{ \sup_{n > 2^p} |\tau_n| > \varepsilon \right\} \leq \sum_{q=p}^{\infty} P\left\{ \max_{2^q < n \leq 2^{q+1}} |\tau_n| > \varepsilon \right\}.
\]

This time we avoid using the maximal inequality (3.3). Instead, we estimate as follows:

\[
\max_{2^q < n \leq 2^{q+1}} |\tau_n| \leq |\xi_{2^q}| + |\tau_{2^q} - \xi_{2^q}| + \max_{2^q < n \leq 2^{q+1}} |\tau_n - \tau_{2^q}|,
\]

whence

\[
(4.7) \quad P\left\{ \max_{2^q < n \leq 2^{q+1}} |\tau_n| > \varepsilon \right\} \leq P\left\{ |\xi_{2^q}| > \frac{\varepsilon}{3} \right\} + P\left\{ |\tau_{2^q} - \xi_{2^q}| > \frac{\varepsilon}{3} \right\}
\]

\[
+ P\left\{ \max_{2^q < n \leq 2^{q+1}} |\tau_n - \tau_{2^q}| > \frac{\varepsilon}{3} \right\}.
\]
According to this, we complete the proof in three steps:

(i) By (4.2) and (4.4),

\[
(4.8) \quad \sum_{q=p}^{\infty} P\left(\left|\xi_{2q}\right| > \frac{\varepsilon}{3}\right) \leq \frac{9}{e^2} \sum_{q=p}^{\infty} E\xi_{2q}^2 = O\left(\frac{1}{e^2}\right) \left(\frac{1}{2^{2p}} \sum_{k=1}^{2^p} \sigma_k^2 + \sum_{k=2^p+1}^{\infty} \frac{\sigma_k^2}{k^2}\right).
\]

(ii) Taking into account the representation

\[
\tau_{2^q} - \xi_{2^q} = -\frac{1}{2^{2q}} \sum_{k=2}^{2^q} (k-1) X_k,
\]

via (3.2), (2.2) and (4.2),

\[
E[\tau_{2^q} - \xi_{2^q}]^2 = O\left(\frac{1}{2^{4q}}\right) \sum_{k=2}^{2^q} (k-1)^2 \sigma_k^2 = O\left(\frac{1}{2^{2q}}\right) \sum_{k=2}^{2^q} \sigma_k^2 = O\left(\frac{1}{2^{2q}}\right) E\xi_{2^q}^2.
\]

By this and (4.8),

\[
(4.9) \quad \sum_{q=p}^{\infty} P\left(\left|\tau_{2^q} - \xi_{2^q}\right| > \frac{\varepsilon}{3}\right) \leq \frac{9}{e^2} \sum_{q=p}^{\infty} E[\tau_{2^q} - \xi_{2^q}]^2 = O\left(\frac{1}{e^2}\right) \left(\frac{1}{2^{2p}} \sum_{k=1}^{2^p} \sigma_k^2 + \sum_{k=2^p+1}^{\infty} \frac{\sigma_k^2}{k^2}\right).
\]

(iii) By the Cauchy inequality,

\[
(4.10) \quad \max_{2^q < n \leq 2^{q+1}} |\tau_n - \tau_{2^q}| \leq \sum_{n=2^q+1}^{2^{q+1}} |\tau_n - \tau_{n-1}| \leq \left\{\sum_{n=2^q+1}^{2^{q+1}} n[\tau_n - \tau_{n-1}]^2\right\}^{1/2}.
\]

The representation

\[
\tau_n - \tau_{n-1} = \sum_{k=1}^{n} \left[\frac{(k-1)(2n-1)}{n^2(n-1)^2} - \frac{1}{n(n-1)}\right] X_k
\]

can be easily checked, whence, via (3.2) and (2.2),

\[
E[\tau_n - \tau_{n-1}]^2 = \frac{O\left(1\right)}{n^2(n-1)^2} \sum_{k=1}^{n} \sigma_k^2.
\]

Thus, by (4.2),

\[
\sum_{n=2^q+1}^{2^{q+1}} nE[\tau_n - \tau_{n-1}]^2 = O\left(1\right) \sum_{n=2^q+1}^{2^{q+1}} \frac{1}{n^2(n-1)^2} \sum_{k=1}^{n} \sigma_k^2
\]
\[
= O\left(\frac{1}{2^{2q}}\right) \sum_{k=1}^{2^q} \sigma_k^2 = O\left(\frac{1}{2^{2q}}\right) E\xi_{2^q}^2.
\]

By this, (4.10) and (4.4),

\[
(4.11) \quad \sum_{q=p}^{\infty} P\left(\max_{2^q < n \leq 2^{q+1}} |\tau_n - \tau_{2^q}| > \frac{\varepsilon}{3}\right) = \frac{O\left(1\right)}{e^2} \sum_{q=p}^{\infty} E\xi_{2^{q+1}}^2
\]
\[
= \frac{O\left(1\right)}{e^2} \left(\frac{1}{2^{2p+2}} \sum_{k=1}^{2^p+1} \sigma_k^2 + \sum_{k=2^p+1}^{\infty} \frac{\sigma_k^2}{k^2}\right).
\]

Putting together (4.6)-(4.9) and (4.11) gives (2.4).
5. The case when \( \{ X_k \} \) is not nearly orthogonal. In this concluding section we assume that conditions (1.2) and (2.1) are satisfied, but (2.2) is not, i.e.

\[
\sum_{j=0}^{\infty} \rho(j) = \infty.
\]

We will require, for the sake of simplicity in calculations, that the sequence

\[
\left\{ \frac{1}{n} \sum_{j=0}^{n-1} \rho(j) : n \geq n_0 \right\}
\]

is nonincreasing from some \( n_0 \) on.

It is easy to see that this requirement is equivalent to

\[
\rho(n) \leq \frac{1}{n} \sum_{j=0}^{n-1} \rho(j) \quad (n \geq n_0).
\]

The following three theorems can be proved by using methods similar to those in §4, and using Lemmas 4 and 5 below.

**Theorem 5.** If \( \{ X_k \} \) is a sequence of rv's satisfying (1.2), (2.1) and (5.2), then the condition

\[
\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} \left( \sum_{j=0}^{k-1} \rho(j) \right) \left[ \log(k + 1) \right]^2 < \infty \quad \rho(0) = 1
\]

implies (1.4).

If the divergence in (5.1) is “fast enough” in the sense that there exist a number \( r > 1 \) and a positive integer \( p_0 \) such that

\[
\frac{\sum_{j=0}^{2^{p_0} - 1} \rho(j)}{\sum_{j=0}^{2^{p_0} - 1} \rho(j)} \geq r \quad (p \geq p_0; \rho(0) = 1),
\]

then the factor \( \log(k + 1)^2 \) in (5.3) becomes superfluous.

**Theorem 6.** If \( \{ X_k \} \) is a sequence of rv's satisfying (1.2), (2.1), (5.2) and (5.4), then the condition

\[
\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} \left( \sum_{j=0}^{k-1} \rho(j) \right) < \infty
\]

implies (1.4).

**Theorem 7.** If \( \{ X_k \} \) is a sequence of rv's satisfying (1.2), (2.1) and (5.2), then condition (5.5) implies (1.6).

Here we present only two lemmas, without entering into details. The first of them is a special case of the maximal inequality [1, Theorem 4].

**Lemma 4.** If \( \{ X_k \} \) is a sequence of rv's such that conditions (3.1) and (5.4) are satisfied, then

\[
E \left[ \max_{1 \leq m \leq n} \left| \sum_{k=a+1}^{a+m} X_k \right| \right]^2 = O \left( \sum_{j=0}^{n-1} \rho(j) \right) \sum_{k=a+1}^{a+n} \sigma_k^2 \quad (a \geq 0, n \geq 1).
\]
This lemma is crucial in the proof of Theorem 6. The second lemma is concerned with numerical series and can be easily checked.

**Lemma 5.** If condition (5.2) is satisfied, then

\[
\sum_{p: 2^p \geq k} \frac{1}{2^{2ap}} \sum_{j=0}^{2^p-1} \rho(j) = O\left( \frac{1}{k^{2a}} \sum_{j=0}^{k-1} \rho(j) \right) \quad (\alpha = 1 \text{ and } 2)
\]

and

\[
\sum_{n=k}^{\infty} \frac{1}{n^3} \sum_{j=0}^{n-1} \rho(j) = O\left( \frac{1}{k^2} \sum_{j=0}^{k} \rho(j) \right) \quad (k \geq 1).
\]

**References**


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