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THE FIRST COEFFICIENT OF THE CONWAY POLYNOMIAL

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ABSTRACT. A formula is given for the first coefficient of the Conway polynomial of a link in terms of its linking numbers. A graphical interpretation of this formula is also given.

Introduction. Suppose that L is an oriented link of n components in S^3. Associated to L is its Conway polynomial $\nabla_L(z)$, which must be of the form

$$\nabla_L(z) = z^{n-1}\left[a_0 + a_1 z^2 + \cdots + a_m z^{2m}\right].$$

Let $\tilde{\nabla}_L(z) = \nabla_L(z)/z^{n-1}$. In this paper we shall give a formula for $a_0 = \tilde{\nabla}_L(0)$ which depends only on the linking numbers of L. We will also give a graphical interpretation of this formula.

It should be noted that the formula we give was previously shown to be true up to absolute value in **[3]**. The author wishes to thank Hitoshi Murakami for bringing Professor Hosakawa's paper to his attention.

We shall assume a basic familiarity with the Conway polynomial and its properties. The reader is referred to **[1, 2, 4, 5** and **6]** for a more detailed exposition. The fact that $\nabla_L(z)$ has the form described above can be found in **[4** or **6]**, for example.

1. A formula for $\tilde{\nabla}_L(0)$. Suppose $L = \{K_1, K_2, \ldots, K_n\}$ is an oriented link in S^3. Let $l_{ij} = \operatorname{lk}(K_i, K_j)$ if $i \neq j$ and define $l_{ii} = -\sum_{j=1, j\neq i}^{n} l_{ij}$. Define the linking matrix $\mathscr{L}$, or $\mathscr{L}(L)$, as $\mathscr{L} = (l_{ij})$. Now $\mathscr{L}$ is a symmetric matrix with each row adding to zero. Under these conditions it follows that every cofactor $\mathscr{L}_{ij}$ of $\mathscr{L}$ is the same. (Recall that $\mathscr{L}_{ij} = (-1)^{i+j}\det M_{ij}$, where M_{ij} is the (i, j) minor of $\mathscr{L}$.)

THEOREM 1. *Let L be an oriented link of n components in S^3. Then $\tilde{\nabla}_L(0) = \mathscr{L}_{ij}$, where $\mathscr{L}_{ij}$ is any cofactor of the linking matrix $\mathscr{L}$.*

PROOF. Let F be a Seifert surface for L. We may picture F as shown in Figure 1.1. Let $\{a_i\}$ be the set of generators for $H_1(F)$ shown in the figure and define the Siefert matrix $V = (v_{ij})$ in the usual way. Namely, $v_{ij} = \operatorname{lk}(a_i^+, a_j)$, where a_i^+ is obtained by lifting a_i slightly off of F in the positive direction. Then if $a_i \cap a_j = \varnothing$ we have $v_{ij} = v_{ji} = \operatorname{lk}(a_i, a_j)$. If $a_i \cap a_j \neq \varnothing$, then $\{i, j\} = \{2k-1, 2k\}$ for some

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$1 \leqslant k \leqslant h$ and $v_{2k-1,2k} = v_{2k,2k-1} - 1$. Hence V is of the form

$$V = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where A is a $2h \times 2h$ matrix and C is a symmetric $(n-1) \times (n-1)$ matrix.

Now a_{2h+i-1} is parallel to K_i for $i > 1$. Hence $v_{2h+i-1,2h+j-1} = l_{ij}$ for $i \neq j$ and $i, j > 1$. Furthermore,

$$\begin{aligned} l_{1i} &= -\sum_{j=2}^{n} v_{2h+i-1,2h+j-1} \\ &= -(l_{i2} + l_{i3} + \cdots + l_{i,i-1} + v_{2h+i-1,2h+i-1} + l_{i,i+1} + \cdots + l_{i,n}). \end{aligned}$$

Therefore, we have

$$v_{2h+i-1,2h+i-1} = -\sum_{j=1, j\neq i}^{n} l_{ij} = l_{i,i}.$$

Hence we have that C is the $(1,1)$ minor of $\mathscr{L}$.

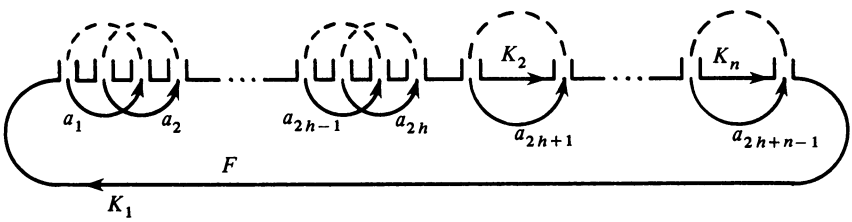

FIGURE 1.1

Now the Conway polynomial can be defined as $\nabla_L(z) = \det(tV - t^{-1}V^T)$, where the right-hand side of this equation is a polynomial in $z = t - t^{-1}$. Hence we have

$$\begin{aligned} \tilde{\nabla}_L(t - t^{-1}) &= \det(tV - t^{-1}V^T)/(t - t^{-1})^{n-1} \\ &= \det\begin{pmatrix} tA - t^{-1}A^T & (t - t^{-1})B \\ (t - t^{-1})B^T & (t - t^{-1})C \end{pmatrix} \Big/ (t - t^{-1})^{n-1} \\ &= \det\begin{pmatrix} tA - t^{-1}A^T & (t - t^{-1})B \\ B^T & C \end{pmatrix}. \end{aligned}$$

So,

$$\tilde{\nabla}_L(0) = \det\begin{pmatrix} A - A^T & 0 \\ B^T & C \end{pmatrix}.$$

But

$$A - A^T = \begin{bmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & 0 & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & 0 & & & 0 & -1 \\ & & & & & 1 & 0 \end{bmatrix}$$

so that $\tilde{\nabla}_L(0) = \det C = \mathscr{L}_{1,1}$. Since all the cofactors of $\mathscr{L}$ are equal, the theorem follows. □

2. A graphical interpretation of $\tilde{\nabla}_L(0)$. Let $\Gamma(L)$ be the complete graph with n vertices. Label the vertices $K_1, \ldots, K_n$ and label the edge connecting K_i and K_j with their linking number l_{ij}. Let G be the set of all subgraphs of Γ consisting of $n-1$ distinct edges together with their vertices. Let T be the subset of G consisting of those graphs which are trees. If $g \in G$ let $\bar{g}$ be the product of the $n-1$ linking numbers associated to the edges of g.

THEOREM 2. *Suppose L is an oriented link in S^3 with n components. Then* $\tilde{\nabla}_L(0) = (-1)^{n-1}\Sigma_{g\in T}\bar{g}$.

PROOF. It follows from Theorem 1 that $\tilde{\nabla}_L(0)$ is a finite sum of terms, where each term is a product of $n-1$ linking numbers together with some integer coefficient. Now each term is actually the product of $n-1$ distinct linking numbers. For consider some l_{ij}. It appears in only four entries of $\mathscr{L}$, namely l_{ii}, l_{ij}, l_{ji}, and l_{jj}. Hence l_{ij} appears only once in the (i, i) minor of $\mathscr{L}$ and so cannot appear to any power greater than one in $\mathscr{L}_{ii}$. Thus we have shown that

$$\tilde{\nabla}_L(0) = \mathscr{L}_{ij} = (-1)^{n-1} \sum_{g\in G} \varepsilon(g)\bar{g}, \tag{2.1}$$

where $\varepsilon(g)$ is some integer.

We want to show that $\varepsilon(g)$ is one if g is a tree and zero otherwise.

Let $\mathscr{L}^q$ be the matrix obtained from $\mathscr{L}^g$ by setting each l_{ij} equal to 1 or 0 depending on whether l_{ij} is associated to g or not. Furthermore, let L^g be any link having $\mathscr{L}^g$ as its linking matrix. Now it follows from (2.1) that $(-1)^{n-1}\varepsilon(g) = \tilde{\nabla}_{L^g}(0)$.

Now suppose that g is not a tree. Then g is either disconnected or misses a vertex of Γ. For suppose that g is connected but is not a tree. Then g contains some loop. This loop has an equal number of edges and vertices. Adding the remaining edges of g cannot increase the number of vertices beyond the number of edges. Hence g has at most $n-1$ vertices since it has $n-1$ edges. Therefore we may choose a split link L^g with linking matrix $\mathscr{L}^g$. But the Conway polynomial of a split link is zero and hence $\varepsilon(g) = 0$.

Thus it only remains to show that $\varepsilon(g) = 1$ if g is a tree. We shall do this by inducting on n. If $n = 2$ it is shown in [**4**] that $\tilde{\nabla}_L(0) = -l_{12}$.[1] This starts the induction. Now suppose that L has n components and that the theorem is true for links with fewer components. Since g is a tree, there is some outermost vertex, say K_i, which is connected by an outermost edge to K_j. Now choose L^g so that it appears in part as shown in Figure 2.1. Changing and smoothing the indicated crossing as illustrated in the figure gives $\nabla_{L^g}(z) = -z\nabla_{L'}(z)$ and hence $\tilde{\nabla}_{L^g}(0) = -\tilde{\nabla}_{L'}(0)$.

[1] Note that a slightly different definition of $\nabla_L(z)$ is used in that paper than here; namely that, $\nabla_L(z) = \det(t^{-1}V - tV^T)$ with $z = t - t^{-1}$.

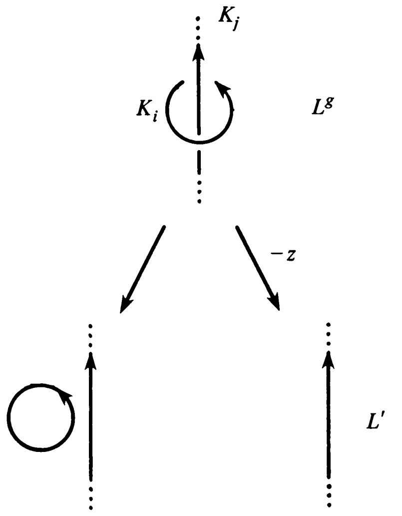

FIGURE 2.1

But L' has $n-1$ components and so by our inductive hypothesis, and the fact that $\Gamma(L')$ has only one subtree h for which $\bar{h} \neq 0$, we have $\tilde{\nabla}_{L^g}(0) = (-1)^{n-1}$. Hence $\varepsilon(g) = 1$. □

As a final remark, note that the number of terms in $\sum_{g \in T} \bar{g}$ is given by $(-1)^{n-1}\bar{\mathscr{L}}_{ij}$, where $\bar{\mathscr{L}}_{ij}$ is the (i, j) cofactor of the matrix $\bar{\mathscr{L}}$ which is obtained from $\mathscr{L}$ by setting the linking numbers equal to 1. It can easily be shown that this number is n^{n-2}.

REFERENCES

1. J. H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra, Pergamon Press, New York, 1970, pp. 329–358.
2. C. Giller, *A family of links and the Conway calculus*, Trans. Amer. Math. Soc. **270** (1982), 75–109.
3. F. Hosokawa, *On ∇-polynomials of links*, Osaka Math. J. **10** (1958), 273–282.
4. J. Hoste, *The Arf invariant of a totally proper link*, Topology Appl. **18** (1984), 163–177.
5. L. H. Kauffman, *The Conway polynomial*, Topology **20** (1981), 101–108.
6. H. Murakami, *The Arf invariant and the Conway polynomial of a link*, Math. Seminar Notes, Vol. 11, Kobe University, 1983, pp. 335–344.

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