CONTINUITY OF THE INVERSE
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ABSTRACT. We present a simple device for proving the continuity of the inverse in a group with a locally Čech-complete topology which makes the multiplication continuous; our proof even works in case the topology is regular and locally strongly countably complete.

We denote by $X$ a group with Hausdorff topology such that the multiplication $X \times X \to X$ is continuous. The following propositions hold:

(A) *If $X$ is locally compact, then $X$ is a topological group* [2].

(B) *If $X$ is completely metrizable, then $X$ is a topological group* [7].

(C) *If $X$ is locally Čech-complete, then $X$ is a topological group* [1].

It is well known that locally compact and completely metrizable spaces are Čech-complete ([4, 3.9.1 and 4.3.26], respectively), so Proposition (C) simultaneously generalizes Propositions (A) and (B). We shall give new and short proofs of these propositions.

Let $e$ be the neutral element of $X$, $\mathcal{U}$ the filter of $e$-neighborhoods; $X$ is a topological group, if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ with $V^{-1} \subset U$. Our proofs are based on the following

**LEMMA.** Let $X$ be regular.

(a) For $U \in \mathcal{U}$ there exists a sequence $(U_n)_{n \in \mathbb{N}}$ in $\mathcal{U}$ such that

$$U_1^2 \subset U, \quad U_{n+1}^2 \subset U_n \quad \text{for } n \in \mathbb{N}.$$ 

(b) If $(U_n)_{n \in \mathbb{N}}$ is chosen as in (a), $x_n \in U_n$, $y_k := x_1 \cdots x_k$, and $(y_k)_{k \in \mathbb{N}}$ has a clusterpoint, then for every $n \in \mathbb{N}$ there exists $k > n$ such that $x_k^{-1} \in U_n$.

**PROOF** (a) This clearly follows from the continuity of the multiplication in $(e,e)$ and the regularity of $X$.

(b) Let $y$ be a clusterpoint of $(y_k)_{k \in \mathbb{N}}$, $n \in \mathbb{N}$. Since $yU_{n+1}$ is a neighborhood of $y$, there exists $k > n + 1$ such that $y_{k-1} \in yU_{n+1}$; and then

$$x_k^{-1} = y_k^{-1}y_{k-1} \in y_k^{-1}yU_{n+1}.$$ 

But $y_k^{-1}y$ is a clusterpoint of $(y_k^{-1}y_j)_{j \in \mathbb{N}}$, and for $j > k$ we have

$$y_k^{-1}y_j = x_{k+1} \cdots x_j \in U_{k+1} \cdots U_j \subset U_k,$$

which implies

$$y_k^{-1}y \in U_k \subset U_{k-1}.$$
From (1) and (2) we get $x_{k-1}^{-1} \in U_{k-1}U_{n+1} \subset U_{n+1}^2 \subset U_n$.

Now the proofs of the propositions. For $U \in \mathcal{U}$ we chose a sequence $(U_n)_{n \in \mathbb{N}}$ as in the lemma and show by contradiction that there is an $n \in \mathbb{N}$ with $U_{n-1} \subset U$.

**Proof of (A).** We may assume that $U$ is compact. Supposing that $U_{n-1} \not\subset U$ for every $n \in \mathbb{N}$ we chose $x_n \in U_n$ with $x_{n-1}^{-1} \not\subset U$. Since $y_k = x_1 \cdots x_k \in U_1 \cdots U_k \subset U$ for every $k \in \mathbb{N}$, and $U$ is countably compact, the sequence $(y_k)_{k \in \mathbb{N}}$ has a clusterpoint; so $x_{n-1}^{-1} \in U$ for some $n \in \mathbb{N}$ by the lemma, a contradiction.

**Proof of (B).** Let the topology of $X$ be given by the complete metric $d$; we may assume that $(U_n)_{n \in \mathbb{N}}$ is a basis of $\mathcal{U}$. Supposing that $U_{n-1} \not\subset U$ for every $n \in \mathbb{N}$ we chose $x_n \in U_n$ with $x_{n-1}^{-1} \not\subset U$. Then $x_n \to e$ and so $xx_n \to e$ for every $x \in X$. Beginning with $\bar{x}_1 := x_1$ we thus inductively find a subsequence $(\bar{x}_n)_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ such that for $\bar{y}_n := \bar{x}_1 \cdots \bar{x}_k$ we have $d(\bar{y}_k, \bar{y}_{k+1}) < 2^{-k}$ for $k \in \mathbb{N}$. Then $(\bar{y}_k)_{k \in \mathbb{N}}$ is convergent. Since $\bar{x}_n \in U_n$, we get $\bar{x}_{n-1}^{-1} \not\subset U$ for some $n \in \mathbb{N}$ from the lemma, a contradiction.

**Proof of (C).** We may assume that $U$ is closed in $X$ and Čech-complete with respect to the induced topology [4, 3.9.6]. There exists a sequence $(\mathcal{A}_n)_{n \in \mathbb{N}}$ of relatively open covers of $U$ such that any family of closed subsets of $U$, which has the finite intersection property and contains sets of diameter less that $\mathcal{A}_n$ for every $n \in \mathbb{N}$, has nonempty intersection [4, 3.9.2]. We may assume that $U_n$ is open and has diameter less than $\mathcal{A}_n$.

Every sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in U_n$ has a clusterpoint in $K := \bigcap_{n=1}^{\infty} U_n$.

Indeed, $F_n := \{x_k : k > n\} \subset U_{n+1}$; since $F_n \subset U_n$ and $U_n$ has diameter less than $\mathcal{A}_n$, we obtain $\emptyset \neq \bigcap_{n=1}^{\infty} F_n \subset K$.

$K$ is a subgroup of $X$.

Indeed, clearly $K^2 \subset K$; we have to show that $K^{-1} \subset K$. For $x \in K$ we have $x_n := x \in U_n$, $n \in \mathbb{N}$, and $y_k = x_1 \cdots x_k = x^k \in K \subset U_k$; from (3) we know that $(y_k)_{k \in \mathbb{N}}$ has a clusterpoint. From our lemma we get $x_{n-1}^{-1} \in U_n$ for every $n \in \mathbb{N}$ and some $k > n$, i.e. $x^{-1} \in K$.

Supposing now that $U_{n-1} \not\subset U$ for every $n \in \mathbb{N}$ we chose $x_n \in U_n$ with $x_{n-1}^{-1} \not\subset U$. From (3) we get a clusterpoint $a \in K$ of $(x_n)_{n \in \mathbb{N}}$. Then $e$ is a clusterpoint of $(z_n)_{n \in \mathbb{N}}$, $z_n := a^{-1}x_{n+1}$; thus every $x \in X$ is a clusterpoint of $(xz_n)_{n \in \mathbb{N}}$; from (4) it is clear that $z_n \in KU_{n+1} \subset U_n$. Beginning with $\bar{z}_1 := z_1$ and $V_1 := U_1$ we inductively find a subsequence $(\bar{z}_n)_{n \in \mathbb{N}}$ of $(z_n)_{n \in \mathbb{N}}$ and a sequence $(V_n)_{n \in \mathbb{N}}$ of open sets such that

$$y_k := \bar{z}_1 \cdots \bar{z}_k \in V_k, V_k \text{ has diameter less than } \mathcal{A}_k,$$

and $\overline{V}_{k+1} \subset V_k$ for every $k \in \mathbb{N}$.

From (5) we obtain a clusterpoint of $(y_k)_{k \in \mathbb{N}}$ in the same way as in the proof of (3). Since $z_n \in U_n$, we get $z_{n-1}^{-1} \in U_1$ for some $k \in \mathbb{N}$ from the lemma. But it is $\bar{z}_k = a^{-1}x_{m+1}$ for some $m \geq k$, and $a^{-1} \in K$ by (4), and thus $x_{m+1}^{-1} = \bar{z}_k^{-1}a^{-1} \in U_1K \subset U_1^2 \subset U$, a contradiction.

**Remarks.** (a) It is well known that in a group with a locally compact or completely metrizable topology which makes the multiplication separately continuous the multiplication is continuous ([3, 6], respectively); it seems to be unknown whether Čech-completeness is sufficient too.
(b) In our proof of Proposition (C) we only used that $X$ is regular and locally strongly countably complete in the sense of [5]; the proof of Brand does not work in this case.

REFERENCES