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THE NUMBER OF T_2-PRECOMPACT GROUP TOPOLOGIES ON FREE GROUPS

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ABSTRACT. It is shown that every free group F admits exactly $2^{2^{|F|}}$ T_2-precompact group topologies. Those topologies can even be chosen to be finer than the finite-index topology on F.

1. Introduction. J. O. Kiltinen has proven in [**12**] that every infinite abelian group G admits $2^{2^{|G|}}$ Hausdorff group topologies, the maximum number possible. K. P. Podewski, in turn, has confirmed this result in [**13**]. Without making use of the quoted result, it is differently proven in [**2, 14, 15**] that, in Kiltinen's statement, "Hausdorff" may be substituted by "T_2-precompact." (A topological group is said to be *T_2-precompact* (by some authors: totally bounded) if it is Hausdorff and precompact with respect to its left (right) uniformity. It is well-known that this property is fulfilled if and only if the topological group is (topologically) isomorphic to a dense subgroup of a compact group.)

The question that suggests itself is whether there exist classes of nonabelian groups for which correspondent statements are valid. The existence of infinite nonabelian groups which are not nondiscretely topologizable to become Hausdorff topological groups has already been established in [**1, 3, 6, 16**]. Pertaining to the raised question, the author has proven that there exist $2^{2^{|F|}}$ Hausdorff group topologies on every free group F [**14**, (5.7), p. 85].

Using only in the abelian case one of the results mentioned above, it is shown in this paper that there exist exactly $2^{2^{|F|}}$ T_2-precompact group topologies on every free group F (Theorem 1). In Theorem 2 one intensifies this result, making use of the methods applied in the proof of Theorem 1. Theorem 2 states that the T_2-precompact group topologies in question can always be chosen to be finer than the *finite-index topology.* (The latter topology is Hausdorff from [**5**, pp. 128–129] and therefore T_2-precompact.)

For further results concerning the topologizability of groups, see [**3**, §9].

2. Notation and conventions. Algebraically isomorphic groups will be identified. Except for this convention, notations from the theory of abelian groups have been taken from [**4**]. In particular, $r(G)$ means the rank of an abelian group G. $|X|$ denotes the cardinality of a set X and τ_X^d the discrete topology on X. Let τ be a topology on X. When possible, (X, τ) will be shortened to X.

Let $(bG, b\tau)$, resp. $b: G \to bG$, be the Bohr-compactification of a topological group (G, τ) or the corresponding Bohr-homomorphism, resp. (cf. [**8**, Chapter V, §4]). Let $G^\frown$ be the character group of a topological group G. For details on Pontrjagin duality see [**7**]. For topological groups G and H, one writes $G \cong H$ if

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some function from G onto H is both a group isomorphism and a homeomorphism. **Z**, **Q**, **R** stand for the integers, the rationals, the reals, respectively. **T** denotes the torus group endowed with the usual compact topology.

The end of each proof is indicated by the symbol $\square$.

3. Results. The proof of Theorem 1 is carried out by using four lemmas. In [**14**] Lemmas 1 and 2 are shown in full detail. But for the reader's convenience, the proofs of the two lemmas will be outlined here.

LEMMA 1 [**14**, (3.9), p. 69]. *Let G be a locally compact abelian group. Then the lattice* $\mathrm{LA}(G)$ *of closed subgroups of G is anti-isomorphic to the lattice* $\mathbf{L}A(G^\frown)$ *of closed subgroups of $G^\frown$.*

PROOF. For $H \in \mathrm{LA}(G)$ let $\mathrm{A}(G^\frown, H)$ be the annihilator of H in $G^\frown$ [**7**, (23.23)]. $\mathrm{A}(G^\frown, H)$ is a closed subgroup of $G^\frown$, because of [**7**, (23.24)(c)]. Now consider the function $F\colon \mathrm{LA}(G) \to \mathrm{LA}(G^\frown)$, defined by $F(H) = \mathrm{A}(G^\frown, H)$ for every $H \in \mathrm{LA}(G)$. Making use of [**7**, (24.9), (24.10)], it is easy to see that F is an anti-isomorphism of lattices. $\square$

In the following $\mathrm{PK}(G)$ denotes the complete lattice of (not necessarily Hausdorff) precompact group topologies on a group G, $\mathrm{LA}(bG, b\tau_G^d)$ means the lattice of closed normal subgroups of $(bG, b\tau_G^d)$, the Bohr-compactification of (G, τ_G^d), and e is the identity of bG.

LEMMA 2 [**14**, (3.7), p. 67]. *For every group G there exists a function* $\Phi'\colon \mathrm{PK}(G) \to \mathrm{LA}(bG, b\tau_G^d)$ *which is an anti-isomorphism of lattices. If (G, τ_G^d) is maximally almost periodic, then Φ' maps the T_2-precompact group topologies on those closed normal subgroups N of bG, satisfying $b(G) \cap N = \{e\}$.*

PROOF. Recall that only a sketch of the proof will be given. In [**14**, (3.6), p. 63] the following Proposition $(*)$ is shown:

PROPOSITION $(*)$. *Let H be a group and τ_1, τ_2 group topologies on H with $\tau_2 \subseteq \tau_1$. Furthermore, let $(bH, b\tau_1)$ be the Bohr-compactification of (H, τ_1) and $b\colon H \to bH$ the corresponding Bohr-homomorphism. Then there exists exactly one closed normal subgroup N of bH in such a way that $(bH/N, b\tau_q^N)$ is the Bohr-compactification of (H, τ_2) and $\pi_N \circ b$ the corresponding Bohr-homomorphism—$b\tau_q^N$ denoting the quotient topology on bH/N, and $\pi_N\colon bH \to bH/N$ being the canonical epimorphism. If (H, τ_1) is maximally almost periodic, then (H, τ_2) is maximally almost periodic if and only if the condition $b(H) \cap N = \{e_{bH}\}$ holds—e_{bH} denoting the identity of bH.*

Now let τ be a group topology on G. By applying Proposition $(*)$ for $H = G$, $\tau_1 = \tau_G^d$, and $\tau_2 = \tau$ one gets exactly one closed normal subgroup $\Phi(\tau)$ of $(bG, b\tau_G^d)$ with the property that $bG/\Phi(\tau)$, endowed with the quotient topology, is just the Bohr-compactification of (G, τ). Thus one has defined a function Φ between the lattice of group topologies on G and the lattice $\mathrm{LA}(bG, b\tau_G^d)$.

Recall that b denotes the Bohr-homomorphism from (G, τ_G^d) into $(bG, b\tau_G^d)$. For $K \in \mathrm{LA}(bG, b\tau_G^d)$ let $\Psi(K)$ be the initial topology on G in respect of the homomorphism $\pi_K \circ b$ from G into bG/K, furnished with the quotient topology, where $\pi_K\colon bG \to bG/K$ means the canonical epimorphism.

Let Φ' be the restriction of Φ to the lattice $\mathrm{PK}(G)$. Then it is proven in [**14**, pp. 65–67] that (Φ', Ψ) is a Galois connection between the complete lattices $\mathrm{PK}(G)$ and $\mathrm{LA}(bG, b\tau_G^d)$, which are also the correspondent sets of closed elements. Using a well-known result about Galois connections, one gets the first part of the assertion. Finally, apply the last part of Proposition $(*)$ to complete the proof. □

From [**9**, Proposition 2.1] one immediately gets

LEMMA 3. *Let G be a compact group, $Z_0(G)$ the component of the identity in the center of G, G' the closure of the commutator group of G and C the component of zero in G/G'. Then there exists a continuous epimorphism $f\colon Z_0G \to C$.*

From [**8**, Satz 5.4.3, p. 128] one easily derives

LEMMA 4. *Let G be a topological group, bG its Bohr-compactification and G', resp. $b'G$, the closure of the commutator groups of G, resp. bG. $b(G/G')$ denotes the Bohr-compactification of G/G'. Then $b(G/G') \cong bG/b'G$.*

THEOREM 1. *Every free group F admits $2^{2^{|F|}}$ T_2-precompact group topologies.*

PROOF. Let m be the free rank of F. The corresponding free abelian group is $F_a := \bigoplus_m \mathbf{Z}$. Knowing $\mathbf{Z}^\frown = \mathbf{T}$ and using [**7**, (23.22)], one gets $\widehat{F_a} \cong \mathbf{T}^m$. ($\mathbf{Z}$ and F_a are endowed with the discrete topology.) From $\mathbf{T} = \mathbf{Q}/\mathbf{Z} \oplus \mathbf{R}$ (cf. [**4**, p. 105]) one derives that $H := \mathbf{R}^m$ is a torsion-free, divisible subgroup of $\widehat{F_a}$. Now [**4**, p. 105] and $r(H) = |H|$ (cf. [**14**, (2.16), p. 45]), i.e. $r(H) = 2^{|F|}$, imply $H = \bigoplus_{2^{|F|}} \mathbf{Q}$. By providing H and $\widehat{F_a}$ with the discrete topology, one gets from [**7**, (24.11)] that $H^\frown$ is topologically isomorphic to a quotient group U of $(\widehat{F_a})^\frown$. Now [**10**, p. 36] assures that $(\widehat{F_a})^\frown$ is the Bohr-compactification bF_a of F_a. Moreover, $U \cong (\Sigma_{\mathbf{a}})^{2^{|F|}}$ from [**7**, (23.22), (25.4)], where $\Sigma_{\mathbf{a}}$ denotes the $\mathbf{a}$-adic solenoid for $\mathbf{a} = (2, 3, \ldots)$—see [**7**]. Since $\Sigma_{\mathbf{a}}$ is connected, one infers that U is also connected. As U is a quotient group of bF_a, there exists a continuous open epimorphism $g\colon bF_a \to U$. Let C denote the component of zero in bF_a. Then [**7**, (7.12)] and the compactness of bF_a imply $g(C) = U$.

That $U^\frown \cong H$ follows from the duality theorem. An abelian group of infinite rank r possesses exactly 2^r subgroups [**4**, Exercise 8, p. 86]. As $r(H) = 2^{|F|}$, $U^\frown$ owns $2^{2^{|F|}}$ subgroups. From Lemma 1 derives that U has $2^{2^{|F|}}$ closed subgroups; using $g(C) = U$ the same is true for C.

By providing F with the discrete topology, Lemma 4 implies $bF_a \cong bF/b'F$. Let Z_0bF be the component of the identity in the center ZbF of bF. Then, because of Lemma 3, there exists a continuous epimorphism $f\colon Z_0bF \to C$. Therefore, Z_0bF has at least $2^{2^{|F|}}$ closed subgroups. As $Z_0bF \subseteq ZbF$, those are closed normal subgroups of bF in addition (Z_0bF is closed in ZbF, which is closed in bF).

One may assume that $m > 1$. (In the case $m = 1$ the result is known because $F = \mathbf{Z}$.) Since (F, τ_F^d) is maximally almost periodic (cf. [**8**, p. 180]) and the center of F is trivial, Lemma 2 applies to give the assertion. □

In Lemma 2 the finite-index topology τ on a group G is mapped to a closed normal subgroup $N := \Phi'(\tau)$ of $(bG, b\tau_G^d)$. One has

LEMMA 5. *N is the component of the identity in $(bG, b\tau_G^d)$.*

PROOF. Following the construction in [**14**], $(bG/N, b\tau_q^N)$ is—$b\tau_q^N$ denoting the correspondent quotient topology—the Bohr-compactification and thus the Hausdorff-completion of (G, τ). τ and hence $b\tau_q^N$ is a linear topology. (A group topology τ on a group G is called *linear* if there exists a fundamental system of neighbourhoods of the identity for τ consisting of subgroups of G.) Therefore, $(bG/N, b\tau_q^N)$ is totally disconnected. Let C denote the component of the identity in $(bG, b\tau_G^d)$. Then from [**7**, (7.12)], one infers $C \subseteq N$.

[**7**, (7.3)] implies that bG/C is totally disconnected in the quotient topology $b\tau_q^C$. Consequently, $b\tau_q^C$ is linear, using [**7**, (7.7)]. Lemma 2 gives a precompact group topology μ on G corresponding to C. $(bG/C, b\tau_q^C)$ represents the Bohr-compactification and therefore also the Hausdorff-completion of (G, μ). Hence μ is linear and thus $\mu \subseteq \tau$. Lemma 2 implies $C \supseteq N$, finally $C = N$. □

Now one can prove

THEOREM 2. *Every free group F admits $2^{2^{|F|}}$ T_2-precompact group topologies finer than the finite-index topology on F.*

PROOF. Notations are as in Theorem 1. Let b_0F denote the component of the identity in bF. In the proof of Theorem 1 it has been shown that $2^{2^{|F|}}$ closed normal subgroups of bF are contained in Z_0bF. Now from $Z_0bF \subseteq b_0F$ one gets the assertion applying Lemmas 2, 5. (Note that the finite-index topology on a free group is Hausdorff.) □

REMARKS. (a) Using results from [**9**], concerning free compact groups, Theorem 1 can be proven for free groups of infinite rank in an alternate way.

(b) The T_2-precompact group topologies from the Theorems 1 and 2 can be considered as being pairwise not topologically isomorphic. This can be shown in the same way as in [**11**, Theorem 2.4].

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