ON A QUESTION OF ARCHANGELSKIJ CONCERNING LINDELÖF SPACES WITH COUNTABLE PSEUDOCHARACTER

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Abstract. We give a negative solution to Archangelskij's problem by showing that there exists a Lindelöf space with countable pseudocharacter which does not admit a continuous one-to-one mapping onto a first countable Hausdorff space.

The aim of this note is to construct, in the usual axioms of set theory, an example mentioned in the abstract (see [Ar, Hypotheses 5.4 and 5.5]). S. Shelah obtained, under some set-theoretical assumptions, a Lindelöf space of cardinality greater than $2^\omega$ with countable pseudocharacter (see [S and HJ]). From the well-known Archangelskij theorem it follows that Shelah's space does not admit a continuous one-to-one mapping onto a first countable Hausdorff space.

Let us denote by $Q$ the set of rational numbers of the unit interval. The symbols $\omega$ and $\omega_1$ stand for the first infinite and first uncountable ordinal numbers respectively.

Example. There is a Lindelöf space $X$ with countable pseudocharacter which does not admit a continuous one-to-one mapping onto a Hausdorff space satisfying the first axiom of countability.

Construction of $X$. There exists a family $\{A_\alpha : 1 \leq \alpha < \omega_1\}$ such that

1. $A_\alpha$ is a countable set consisting of strictly increasing sequences of $Q$ of length $\alpha$ for $1 \leq \alpha < \omega_1$;
2. if $\alpha < \beta < \omega_1$, then $p_\alpha(A_\beta) = A_\alpha$; $p_\alpha$ stands for the projection onto the first $\alpha$ coordinates;
3. if $a \in A_\alpha$ for $1 \leq \alpha < \omega_1$, then for every limit ordinal number $\beta < \alpha$, $a(\beta) = \sup\{a(\lambda) : \lambda < \beta\}$ and $sup\{a(\lambda) : \lambda < \alpha\}$ are rational numbers of $Q$;
4. if $\alpha < \beta < \omega_1$, $a \in A_{\alpha+1}$, $r \in Q$ and $a(\alpha) < r$, then there exists $b \in A_{\beta+1}$ such that $p_{\alpha+1}(b) = a$ and $b(\beta) = r$ (see [J, p. 91, the construction of the Aronszajn tree]).

Let us attach to $a \in A_\alpha$, for $1 \leq \alpha < \omega_1$, $x_a \in Q^{\omega_1}$ defined by

$$x_a(\beta) = \begin{cases} a(\beta), & \text{if } \beta < \alpha, \\ \sup\{a(\lambda) : \lambda < \alpha\}, & \text{if } \beta \geq \alpha. \end{cases}$$

Let $X = \bigcup\{X_\alpha : 1 \leq \alpha < \omega_1\}$, where $X_\alpha = \{x_a : a \in A_\alpha\}$, be a subspace of $Q^{\omega_1}$. In [Al] it was proved that $Y \times X^\omega$ is Lindelöf provided that $Y$ is a hereditarily Lindelöf space. We shall sketch the proof of the Lindelöf property in $X$ for the sake of

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completeness. Let $\mathcal{B}$ be a countable base of $Q$ and $\mathcal{V}$ an open covering of $X$. For $x \in X_\alpha$, $\alpha < \omega_1$, let

$$\mathcal{A}(x) = \left\{ B \in \mathcal{B} : x(\alpha) \in B \text{ and there are } \alpha < \beta(x, B) < \omega_1 \text{ and } V \in \mathcal{V} \text{ such that } F(x, B, \beta(x, B)) = \left( \prod_{\lambda < \omega_1} F_\lambda \right) \cap X \subset V \right\},$$

where

$$F_\lambda = \begin{cases} \{ x(\lambda) \}, & \text{if } \lambda \leq \alpha, \\ B, & \text{if } \lambda = \beta(x, B), \\ Q, & \text{otherwise}. \end{cases}$$

Since $X$ consists of increasing sequences, $\mathcal{A}(x) \neq \emptyset$ for every $x \in X$. Let

$$\beta_1 = \sup \{ \beta(x, B) : x \in X_1 \text{ and } B \in \mathcal{A}(x) \}. $$

Since $X_1$ and $\mathcal{A}(x)$, for $x \in X_1$, are countable sets, $\beta_1 < \omega_1$. If $\beta_n$ is defined, then let

$$\beta_{n+1} = \sup \{ \beta(x, B) : x \in \bigcup \{ X_\lambda : \lambda \leq \beta_n + 1 \} \text{ and } B \in \mathcal{A}(x) \}$$

and

$$\beta = \sup \{ \beta_n : n \in \mathbb{N} \}. $$

To finish the proof of the Lindelöf property of $X$ it is enough to show that

$$X = \bigcup \{ F(x, B, \beta(x, B)) : x \in \bigcup \{ X_\lambda : \lambda < \beta \} \text{ and } B \in \mathcal{A}(x) \}. $$

Let $x$ be an element of $X_\alpha$ for $\alpha \geq \beta$. Then $p_\beta(x)$ belongs to $A_\beta$. Let $x'$ be a point of $X_\beta$ such that $p_\beta(x') = p_\beta(x)$. There exist $B \in \mathcal{B}$, $\alpha_1$ and $\alpha_2$ and $V \in \mathcal{V}$ such that $x'(\beta) \in B$, $\alpha_1 < \beta < \alpha_2$ and $F = (\prod_{\lambda < \omega_1} F_\lambda) \cap X \subset V$, where

$$F_\lambda = \begin{cases} \{ x'(\lambda) \}, & \text{if } \lambda < \alpha_1, \\ B, & \text{if } \lambda = \alpha_2, \\ Q, & \text{otherwise}. \end{cases}$$

Without loss of generality we can assume that $\sup \{ x'(\lambda) : \lambda < \alpha_1 \} \in B$. Let $v$ be an element of $X_\alpha$ such that $p_\beta(v) = p_\beta(x')$. Then $B \in \mathcal{A}(v)$ and $\beta(v, B) < \beta < \alpha_2$. It is easy to see that $x' \in F(v, B, \beta(v, B))$. Since $p_{\beta_1}^{-1} p_\beta(F(v, B, \beta(v, B))) = F(v, B, \beta(v, B))$ and $p_\beta(x') = p_\beta(x)$, $x \in F(v, B, \beta(v, B))$. We conclude that $X$ is a Lindelöf space.

If $x \in X_\alpha$, $\alpha < \omega_1$, then $\{ x' \in X : p_{\alpha+2}(x') = p_{\alpha+2}(x) \} = \{ x \}$ and $p_{\alpha+2}(X)$ is countable, so $\{ x \}$ is a $G_\delta$-subset of $X$.

To finish the proof of the properties of $X$ it is enough to show that $X$ does not admit a weaker Hausdorff topology $\tau$ which satisfies the first axiom of countability. Suppose not and let $\tau$ be a weaker Hausdorff topology on $X$ satisfying the first axiom of countability. If $x \in X$ then there exists $\beta(x) < \omega_1$ such that for every open, in $\tau$, neighbourhood $U$ of $x$ there is a basic open neighbourhood $B(U) = (\prod_{\lambda < \omega_1} B_\lambda(U)) \cap X$ of $x$, with respect to the Tychonoff topology, such that $B_\lambda(U) = Q$ for $\lambda \geq \beta(x)$ and $B(U) \subset U$. The existence of $\beta(x)$ is an immediate consequence of the fact that $\tau$ satisfies the first axiom of countability. Let $\beta_1 = \sup \{ \beta(x) : x \in X_1 \}$. If $\beta_n$ is defined, then let

$$\beta_{n+1} = \sup \{ \beta(x) : x \in \bigcup \{ X_\lambda : \lambda \leq \beta_n + 1 \} \}.$$
and \( \beta = \sup(\beta_n : n \in \mathbb{N}) \). Notice that \( \beta \) is a limit ordinal number less than \( \omega_1 \). Let \( x_1 \) and \( x_2 \) be points of \( X_\beta \) and \( X_{\beta+2} \), respectively, such that \( p_{\beta+1}(x_1) = p_{\beta+1}(x_2) \) and \( x_1(\beta + 1) \neq x_2(\beta + 1) \). To prove that \( \tau \) is not a Hausdorff topology it is enough to show that if \( U \) is an open neighbourhood of \( x_1 \), with respect to \( \tau \), then there is a sequence \( (y_n)_{n=1}^{\infty} \) of points of \( U \) converging to \( x_2 \), with respect to the Tychonoff topology. Let \( B \in \mathcal{B} \) and \( \alpha_1, \alpha_2 < \omega_1 \) be such that \( \beta_1 < \alpha_1 < \beta < \alpha_2 \), \( x_1(\lambda) \in B \) if \( \lambda \geq \alpha_1 \) and \( F = (\Pi_{\lambda<\omega_1}F_\lambda) \cap X \subset U \), where

\[
F_\lambda = \begin{cases} 
\{ x_1(\lambda) \}, & \text{if } \lambda \leq \alpha_1, \\
B, & \text{if } \lambda = \alpha_2, \\
Q, & \text{otherwise}.
\end{cases}
\]

Let \( z_n \) be a point of \( X_{\alpha_n+1} \), where \( \alpha_n = \max(\beta_n, \alpha_1) \), such that \( p_{\alpha_n+1}(z_n) = p_{\alpha_n+1}(x_1) \). Since \( \beta \) is a limit ordinal number, \( \alpha_n + 1 < \beta \). By the definition of \( \beta(z_n) \), there exists a basic open neighbourhood \( G(z_n) = (\Pi_{\lambda<\omega_1}G_\lambda(z_n)) \cap X \) of \( z_n \), with respect to Tychonoff topology, such that \( G_\lambda(z_n) = Q \), if \( \lambda \geq \beta(z_n) \) and \( G(z_n) \subset U \). Notice that \( \beta(z_n) < \beta \) for \( n \in \mathbb{N} \). Let \( y_n \) be an element of \( G(z_n) \cap X_{\beta+2} \) such that \( p_{\alpha_n+1}(z_n) = p_{\alpha_n+1}(y_n) \) and \( y_n(\lambda) = x_2(\lambda) \) for \( \lambda \geq \beta \). The existence of \( y_n \) is an easy consequence of (4), \( \beta(z_n) < \beta \), \( z_n(\lambda) \leq x_1(\lambda) \), for \( \lambda < \omega_1 \), \( p_{\beta+1}(x_1) = p_{\beta+1}(x_2) \), and \( x_1(\beta + 1) < x_2(\beta + 1) \). If \( G = (\Pi_{\lambda<\omega_1}G_\lambda) \cap X \) is a basic open neighbourhood of \( x_2 \), in the Tychonoff topology, \( \alpha = \sup(\lambda < \beta : G_\lambda \neq Q) \) and \( k \) is such that \( \beta_k > \alpha \), then \( y_n \in G \) provided that \( n \geq k \), so we conclude that \( (y_n)_{n=1}^{\infty} \) converges to \( x_2 \) in the Tychonoff topology.

**Remark.** Let \( Z \) be a subspace of \( I^{\omega_1} \), where \( I \) stands for the unit interval, of all points of \( I \) satisfying the following conditions:

(i) for every \( \epsilon > 0 \) and \( z \in Z \), \( \{ \alpha < \omega_1 : z(\alpha) > \epsilon \} \) is finite;

(ii) for every \( z \in Z \), \( \{ \alpha < \omega_1 : z(\alpha) > 0 \} \) is an initial interval of \( \omega_1 \).

It is easy to see that \( Z \) has countable pseudocharacter. In [C] it was proved that \( Z \) has the Lindelöf property. Using our method one can show that \( Z \) does not admit a continuous one-to-one mapping onto a first countable Hausdorff space.

**References**


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