THE POTENTIAL J-RELATION AND AMALGAMATION BASES FOR FINITE SEMIGROUPS

T. E. HALL AND MOHAN S. PUTCHA

ABSTRACT. Let $S$ be a finite semigroup, $a, b \in S$. When does there exist a finite semigroup $T$ containing $S$ such that $a \mathrel{J} b$ in $T$? This problem was posed to the second named author by John Rhodes in 1974. We show here that if $a, b$ are regular, then such a semigroup $T$ exists if and only if either $a \mathrel{J} b$ in $S$, or $a \notin SbS$ and $b \notin SaS$. We use this result to show that amalgamation bases for the class of finite semigroups have linearly ordered $J$-classes.

1. Preliminaries. For $X$ any set, $|X|$ denotes the cardinality of $X$ and $\mathcal{T}(X)$ denotes the full transformation semigroup on $X$, acting on the right on $X$. If $a \in \mathcal{T}(X)$, then $\rho(a)$ denotes the rank of $a$, namely $|X^a|$. Let $F$ be a field. Then we let $M_n(F)$ denote the multiplicative monoid of all $n \times n$ matrices over $F$. If $a \in M_n(F)$, then $\rho(a)$ denotes the rank of $a$. Let $X$ be a set, with $|X| = n$. Then $\mathcal{T}(X)$ acts naturally on a vector space of dimension $n$ over $F$. Thus $\mathcal{T}(X)$ embeds naturally into $M_n(F)$. Moreover, this embedding preserves rank.

Let $S$ be a semigroup. If $a, b \in S$, then we write $a \mathrel{J} b$ (a divides b) if $b \in S^1aS^1$, i.e., if $a \mathrel{J} b \mathrel{J} a$.

By an amalgam we mean a list $(S, T; U)$ of semigroups such that $S \cap T = U$, and we say the amalgam is embeddable if there is a semigroup $W$ with $S$ and $T$ as subsemigroups (see [5] for more details). By an amalgamation base for a class $\mathcal{C}$ of semigroups we mean any $U \in \mathcal{C}$ such that every amalgam $(S, T; U)$ with $S, T \in \mathcal{C}$ is embeddable in some $W \in \mathcal{C}$ (see [4] for some examples).

2. The potential J-relation.

THEOREM 1. Let $S$ be a finite semigroup, $A \subseteq S$.

(a) If for any $a, b \in A$, either $a \mathrel{J} b$ or $a \mathrel{J} b$, then there exists a finite semigroup $T$ containing $S$ such that $A$ lies in a $J$-class of $T$. If $S$ is an inverse semigroup, then $T$ can be chosen to be an inverse semigroup.

(b) Conversely, if $S$ can be embedded in a finite semigroup $T$ containing $A$ within a $J$-class and if every element of $A$ is regular in $S$, then for any $a, b \in A$, either $a \mathrel{J} b$ in $S$ or $a \mathrel{J} b$, $b \mathrel{J} a$ in $S$.

PROOF. (a) Without loss of generality we can assume that $S = S^1$ and that $A$ contains no pair of $J$-equivalent elements. We shall embed $S$ in a transformation semigroup $\mathcal{T}(X)$ such that the elements of $A$ all have the same rank, i.e., are $J$-related.

Choose $a \in A$ such that $|Sa|$ is the maximum. Let $A = \{a = a_0, a_1, \ldots, a_p\}$. Let $I_j = \bigcup_{k \neq j} Sa_k S$, $j = 1, \ldots, p$. Let $S_0 = S$, $S_j = S/I_j$, $j = 1, \ldots, p$. Then $S$ acts on the right on $S_i$, $i = 0, \ldots, p$. Let $|Sa_i| = m_i$, $i = 0, \ldots, p$, $\beta_j = |S_j a_j| > 1$,
$j = 1, \ldots, p$, $\alpha_0 = \prod_{j=1}^p (\beta_j - 1)$ and $\alpha_j = \alpha_0 (m_0 - m_j)/(\beta_j - 1)$, $j = 1, \ldots, p$. So $\alpha_0 \geq 1$, $\alpha_j \geq 0$, $j = 1, \ldots, p$. Let $X$ denote the disjoint union of $\alpha_i$ copies of $S_i$, $i = 0, \ldots, p$. Then $S$ acts faithfully on the right on $X$ and thus $S$ embeds in $T(X)$. Clearly $|Xa_0| = \alpha_0 m_0 + \sum_{j=1}^p \alpha_j$, and $|Xa_j| = \alpha_0 m_j + \alpha_j \beta_j + \sum_{k \neq 0, j} \alpha_k$, $j = 1, \ldots, p$. Routine calculations now show that $|Xa_0| = |Xa_j|$, $j = 1, \ldots, p$. So $A$ lies within a $J$-class of $T(X)$.

If $S$ is an inverse semigroup, then we can use the Preston-Vagner representation of $S$ and the $S_j$'s to obtain a representation of $S$ in the symmetric inverse semigroup on $X$.

(b) Suppose that $S$ can be so embedded in $T$ and suppose, to the contrary, that there exist $a, b \in A$ such that $a \preceq b$ and $b \preceq a$. By [2, Theorem 1 or 7, Proposition 3.1] there exist idempotents $e \in J_a$, $f \in J_b$ such that $e > f$. Clearly $e J f$ in $T$, so $T$ contains a copy of the bicyclic semigroup [1, Theorem 2.54] and hence is infinite, a contradiction.

The next theorem shows that with respect to semigroup division, the whole semigroup can be put into a $J$-class.

**Theorem 2.** Let $S$ be a finite semigroup. Then there exists a finite regular semigroup $T$, a subsemigroup $T_0$ of $T$, a $J$-class $J$ of $T$ and a morphism $\phi : T_0 \to S$ such that $\phi(J \cap T_0) = S$. If $S$ is a regular semigroup, then $T_0$ can be chosen to be a regular semigroup. If $S$ is an inverse semigroup, then $T_0$ and $T$ can be chosen to be inverse semigroups.

**Proof.** Let $T_0 = S \times \{0, 1\}$ with the following multiplication:

$$(a, \alpha)(b, \beta) = (ab, \gamma),$$

where

$$\gamma = \begin{cases} 1 & \text{if } \alpha = \beta = 1 \text{ and } a J b J ab, \\ 0 & \text{otherwise.} \end{cases}$$

It can be verified that $T_0$ is a semigroup. If $S$ is a regular semigroup, then $T_0$ is a regular semigroup. If $S$ is an inverse semigroup, then $T_0$ is an inverse semigroup. The map $a \mapsto (a, 0)$ embeds $S$ into $T_0$.

Easy manipulation (or Remark 1 below) now shows that the subset $A = S \times \{1\}$ satisfies the hypothesis of Theorem 1(a). Thus there exists a finite semigroup $T$ containing $T_0$ such that $A$ lies in a $J$-class $J$ of $T$. the map $\phi : T_0 \to S$ given by $\phi(a, \alpha) = a$ is a morphism and $\phi(J \cap T_0) \supseteq \phi(A) = S$. This proves the theorem.

**Remark 1.** In fact $S \times \{0\}$ is an ideal of $T_0$ and $T_0/(S \times \{0\})$ is isomorphic to the 0-direct union [1, §6.3] of all the principal factors $\{J \cup \{0\} : J \in S/J\}$ (including, if $S = S^0$, the two element semilattice), while $T_0$ is the ideal extension of $S \times \{0\}$ (or $S$) by $T_0/(S \times \{0\})$ determined by the partial morphism $(s, 1) \mapsto (s, 0)$ for $s \in S$.

### 3. Amalgamation bases.

**Theorem 3.** For any amalgamation base of the class of finite [finite regular, finite inverse] semigroups, the $J$-classes are linearly ordered.

**Proof.** Take any amalgamation base, $U$ say, of the class of finite semigroups, and suppose, to the contrary, that there are two elements, $a$ and $b$ say, whose $J$-classes are not comparable.

Case I. $a$ or $b$ is regular, say $a$. Take any idempotent $e \in J_a$. Form a semigroup $U' = U \cup \{e'\}$ containing $U$ as a subsemigroup (where $e' \notin U$) by defining $e'^2 =
AMALGAMATION BASES FOR FINITE SEMIGROUPS

363

e', e' \in U. Then e' b and b e' in U', so by Theorem 1 there exists a finite semigroup S containing U' such that c' \ J b in S.

Since J c = J a and J b are not comparable, again by Theorem 1, there exists a finite semigroup T containing U such that e \ J b in T.

The amalgam (S, T; U) is embeddable in a finite semigroup, W say, since U is an amalgamation base for the class of finite semigroups. But then in W we have e \ J b \ J e' and e < e', whence W contains a copy of the bicyclic semigroup [1, Theorem 2.54] and is infinite, a contradiction.

The proof so far is easily modified to give a proof of the bracketed statements. Note that the result for the class of finite regular semigroups is a trivial corollary of the result for the class of finite semigroups, since a regular semigroup is an amalgamation base for either class if and only if it is one for the other class (since any finite semigroup embeds in a finite regular semigroup).

Case II. a and b are not regular. Without loss of generality we can assume that U is a subsemigroup of M_n(Z_2) for some positive integer n, and that n > \rho(a) \geq \rho(b). Consider the embedding \theta: M_n(Z_2) \rightarrow M_{2n}(Z_2) given by \theta(c) = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}. Put a' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, where 1 denotes the n \times n identity matrix; then \rho(a') = n + \rho(a) > 2\rho(a) = \rho(\theta(a)) \geq \rho(\theta(b)) and a'^2 = \theta(a). (This method of finding square roots is a variation of that due to C. J. Ash [5, Theorem 5.1].) Thus, so far, we have embedded U in a finite semigroup U' with an element a' such that a'^2 = a and b \uparrow a' (note that from b \uparrow a in U we do not get b \uparrow a in U', so it is not immediate from a'^2 = a that b \uparrow a' in U').

Now consider the embedding \psi: U \rightarrow U' \times (U/U1aU1) given by \psi(u) = (u, \phi(u)), where \phi is the canonical morphism of U upon U/U1aU1. Put v = (a', 0); then v^2 = (a, 0) = \psi(a) and v \uparrow (b, \phi(b)) = \psi(b) since \phi(b) \neq 0. Also \psi(b) \uparrow v since b \uparrow a' in U'. Thus we have a semigroup V containing U and an element v such that v^2 = a, v \uparrow b, b \uparrow v.

By Theorem 1, there exists a finite semigroup S containing V such that v \ J b in S. Also, since in U, J a and J b are not comparable, by Theorem 1 there is a finite semigroup T containing U such that a \ J b in T.

Since U is an amalgamation base for the class of finite semigroups, the amalgam (S, T; U) is embeddable in a finite semigroup W, say. Then v \ J b \ J a = v^2 in W, and since W is finite, we have v \uparrow 2 = a in W. Thus a is in a subgroup of W and hence in a subgroup of U, contradicting that a is not regular in U.

REMARK 2. The existence of a finite inverse semigroup which is not an amalgamation base for the class of finite inverse semigroups was first shown by C. J. Ash: his example, given in [3], is the three element semilattice which is not a chain. His construction and proof led us to the proof in Case I above.

REMARK 3. One of the authors has recently shown that the J-classes being linearly ordered is also a sufficient condition for a finite inverse semigroup to be an amalgamation base of the class of finite inverse semigroups.

REFERENCES


DEPARTMENT OF MATHEMATICS, MONASH UNIVERSITY, CLAYTON, VICTORIA 3168, AUSTRALIA

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27695