ON SOME PROPERTIES OF THE BANACH ALGEBRAS $A_p(G)$ FOR LOCALLY COMPACT GROUPS

Dedicated to my teacher Rafael Artzy with gratitude and respect

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Abstract. We strengthen and improve theorems of Choquet-Deny and of Foguel concerning convolution equations and iterates of convolution of a measure to all algebras $A_p(G)$ and all locally compact groups. Furthermore, we improve results of H. P. Rosenthal on ideals of $A(G)$ to the algebras $A_p(G)$ and show that some hold for amenable groups but not for free nonabelian groups. Finally, we improve a (possibly) weak version of a theorem of Gilbert on projections onto some subspaces of $L^\infty(G)$ to all locally compact groups.


Theorem (Foguel). Let $G$ be an l.c.a. group and $\mu \in M(G)$ be such that $\sup_n \|\mu^n\| < \infty$. Then $\lim \|\mu^n \ast f\|_1 = 0$ for each $f \in I_e = \{ f \in L^1(G); f(e) = 0 \}$ if and only if $|\mu(\gamma)| < 1$ for all $\gamma \in \Gamma \setminus \{ e \}$.

Here $\mu^n$ is the $n$-times convolution power of $\mu$ and $e$ denotes the unit of $\Gamma$.

Theorem (Choquet-Deny). Let $G$ be an l.c.a. group and $\mu \in M(G)$. The following are equivalent:

(i) for $f \in L^\infty(G)$, $\mu \ast f = f$ implies $f = \text{constant}$.
(ii) $\mu(\gamma) \neq 1$ for $\gamma \in \Gamma \setminus \{ e \}$.

We strengthen and improve (the dual version of) both results to all locally compact groups and all algebras $A_p(G)$, $B_p^M(G)$, $PM_p(G)$ in the first two theorems of the paper and in the remarks after them. (If $G$ is abelian and $p = 2$, then $A_2(G) = A(G) = L^1(\Gamma)$, $B_2^M(G) = B(G) = M(\Gamma)$, and $PM_2(G) = L^\infty(\Gamma)$.) Furthermore, we point out that if, for some $\lambda \in \mathbb{C}$ and $u \in B_p^M$, $E_\lambda = \{ \phi \in PM_p; u \cdot \phi = \lambda \phi \}$ is a reflexive Banach space and if $G$ is amenable, then $E_\lambda$ is finite dimensional. This is false if $G$ is discrete and contains the free group on two generators (via existence of Leinert sets in $G$).

In Theorem 4 we improve a result of H. P. Rosenthal [18, p. 39] to all amenable groups $G$. We show that, if for some closed ideal $I$, $A_p/I$ is a reflexive Banach space and if $G$ is amenable, then $A_p/I$ is finite dimensional. Again, using Leinert sets, we

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1 Thanks are due to T. Ramsey for providing us with a preprint of [13] and for discussions related to it.
get a counterexample in case G is discrete and contains a free group on two
generators.

In Theorem 5 we improve the following result of H. P. Rosenthal, which is part of
Theorem 2.12 on p. 53 in [18]: If G is any nondiscrete locally compact abelian group,
then any nonzero ideal of A(G) contains an isomorphic copy of l^1. We show that this
result is true for all nondiscrete G and for all algebras A_p(G). This is false if G is
discrete and p = 2, as shown by M. A. Picardello [21].

In Theorem 6 we prove a result related to the beautiful main theorem of J. E.
Gilbert on existence of projections onto w* closed translation invariant subspaces of
L^∞(G). Again this is done in the framework of the algebras A_p, B^M_p, PM_p.

The reader not familiar with the algebras A_p may find the results that follow of
interest even for p = 2 and abelian G.

Definitions and notation. C will denote the complex field. G will always denote a
locally compact group, C_0(G) (C_c(G)) the continuous functions on G, which tend
to 0 at ∞ (with compact support). L^p(G), 1 ≤ p ≤ ∞, will be the usual spaces of
p-integrable functions with respect to a fixed left Haar measure m and ∥f∥_p =
(∫ |f|^p dm)^1/p, ∥f∥_∞ = ess sup |f(x)|. We follow Herz [8] for notation and properties
of the Banach algebras A_p(G) = A_p. One has that A_2(G) = A(G) = the Fourier
algebra of G à la Eymard [2]. We denote by ∥v∥_A_p the norm in A_p (or just ∥v∥
when the context is clear). We denote by B^M_p = B^M_p(G) the set of bounded complex
functions u on G such that w ∈ A_p(G) for all v ∈ A_p. The norm in B^M_p is given by
∥u∥_B^M_p = sup{∥uv∥_A_p; ∥v∥_A_p = 1}. If G is abelian with dual Γ, then A_2(G) = A(G) =
L^1(Γ)∗ and B^M_p(G) = B(G) = M(Γ), where M(Γ) is the Banach algebra of bounded
complex measures on Γ and * denotes Fourier transform. PM_p(G) is the Banach
space dual of A_p as in [8]. If G is abelian, then PM_2(G) = L^∞(Γ). We define the
module action of B^M_p on PM_p by ⟨φ, w⟩ for φ ∈ PM_p, v ∈ A_p, u ∈ B^M_p. If v ∈ A_p, then supp v denotes the closure in G of {x; v(x) ≠ 0}.

If C is a subset of A_p, then C will denote the norm closure of C in A_p.

Some interesting properties of the algebras A_p(G) for abelian G have been

If X is a Banach space, then L(X) will denote the bounded linear operators T:
X → X with ∥T∥ = sup{∥Tx∥; ∥x∥ = 1}. If A, B are subsets of C, then A ~ B will
denote the set-theoretical difference of A and B. And if τ is a topology on C, then
τ cl A will denote the τ-closure of A in C.

The following lemma was obtained independently of the proofs given in [13].

Lemma 1. Let u ∈ B^M_p(G) be such that |u(x)| ≤ 1 for all x and, let L = {x;
|u(x)| = 1}. If v ∈ C_00 ∩ A_p(G) is such that {supp v} ∩ L = ∅, then ∥u^n v∥_A_p → 0
as n → ∞. (L = ∅ is allowed.)

Proof. Let S = supp v. S is compact and there exists a symmetric neighborhood
of e, V such that SV^2 ∩ LV = ∅ and V is compact. For x in G define
g(x) = λ(V)^{-1}[1_{SV} * 1_{V}](x) = λ(V)^{-1}(xV ∩ SV).

Then g ∈ A_p by the definition of A_p, g(x) = 1 on S, g(x) = 0 if x is off SV^2 and
0 ≤ g(x) ≤ 1 for all x. Furthermore, ug ∈ A_p ∩ C_00. Hence |u(x)g(x)| < 1 for all x

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in $G$ (if $x \in L$ then $g(x) = 0$, and if $x \notin L$ then $|u(x)| < 1$, while $0 \leq g \leq 1$). Let $d = \sup\{|u(x)g(x)|; x \in G\} < 1$. The maximal ideal space of $A_p(G)$ is $G$ [8, p. 102], hence the spectral radius of $ug$ is $d = \lim_n\|u^n\|^{1/n} < 1$. Now choose $\delta > 0$ such that $d + \delta < 1$. Then for some $n_0$ we have $\|u^n\|_{A_p} < (d + \delta)^n$ if $n \geq n_0$. Hence $\|u^n\|_{A_p} \to 0$. But $g^n(x) = 1$ if $x \in S$; thus $g^n = v$. It follows that

$$\|u^n v\|_{A_p} = \|u^n g^n v\|_{A_p} \leq \|u^n g^n\|_{A_p} \|v\|_{A_p} \to 0.$$  

**Remark 1.** Note that no assumption on the boundedness of $\|u^n\|_M$ is made in the above lemma. If, however, $\sup\|u^n\| = C < \infty$, then $|u^n(x)| \leq C$ for all $n$; thus $|u(x)| \leq 1$ for all $x$.

Let $L \subset G$ be closed. Let $J_L = \{v \in A_p \cap C_0; \supp v \cap L = \emptyset\}$ and $I_L = \{v \in A_p; v = 0$ on $L\}$. Clearly $J_L \subset I_L$.

**Remark 2.** If $u \in B_p^\infty$ is such that $\|u^n v\| \to 0$ for each $v \in J_L$ (where $L \subset G$ is any closed set), then $|u(x)| < 1$ for each $x \in G \sim L$. Since if $|u(x_0)| = 1$, $x_0 \in G \sim L$, then there is some $v \in J_L$ such that $v(x_0) = 1$. Then $1 = \|u^n(x_0) v(x_0)\| \leq \|u^n v\|$.  

**Theorem 2.** Let $u \in B_p^\infty(G)$ be such that $\sup\|u^n\|_M = C < \infty$, and let $L = \{x; |u(x)| = 1\}$. Then $\|u^n v\|_{A_p} \to 0$ for each $v \in J_L$.  

($\star$) If $L$ is a set of spectral synthesis then $\|u^n v\|_{A_p} \to 0$ for each $v \in I_L$.

**Proof.** Let $v \in J_L$, $\epsilon > 0$. Let $v_0 \in J_L$ be such that $\|v - v_0\| < \epsilon$. Then

$$\|u^n v\| \leq \|u^n (v - v_0)\| + \|u^n v_0\| \leq C\epsilon + \|u^n v_0\| \to C\epsilon$$

by the above lemma. If $L$ has spectral synthesis, then $J_L = I_L$.

**Remarks.** In many cases the condition $\sup\|u^n\|_M < \infty$ forces the set $L$ to be a set of spectral synthesis:

(a) Let $p = 2$ and $G$ abelian. Assume that $u \in B(G) = M(\hat{G})$ is such that $\sup\|u^n\|_B(G) < \infty$. Then $\sup\{|u^n(x); x \in G\} \leq 1$ and $L = \{x; |u(x)| = 1\}$ is a closed subset of the coset ring of $G$ and, as such, is even a strong Ditkin set, by J. E. Gilbert [5, 6] or B. Schreiber [15, Theorem 6.2 and 14, Theorem 2.6]. A fortiori, $L$ is a set of spectral synthesis.

Foguel's result is thus improved in the

**Corollary.** Let $G$ be l.c.a., $\mu \in M(G)$ satisfy $\sup\|\mu^n\| < \infty$, and let $L = \{\gamma \in \Gamma; |\hat{\mu}(\gamma)| = 1\}$. Then $\|\mu^n \ast f\|_1 \to 0$ iff $f \in I_L = \{f \in L^1(G); \hat{f} = 0$ on $L\}$. (Since clearly $|\hat{f}(r)| \leq \|\mu^n \ast f\|_1$ if $r \in L$.)

(b) Let $p = 2$, $G$ arbitrary. Let $B(G)$ be as in [2], and let $u \in B(G) \subset B_2^\infty(G)$ be a positive definite function such that $u(e) = 1$. Then $\|u\|_{B(G)} = 1$ and $L = \{x; |u(x)| = 1\}$ is a closed (not necessarily normal) subgroup of $G$. Then $\|u^n\|_M = \|u^n\|_{B(G)} = 1$ and $L$ has spectral synthesis, as shown by Takesake and Tatsuma in [16]. If $u \in B(G)$ only satisfies $|u(x_0)| = \|u\| = 1$ at some $x_0$ in $G$, then it can easily be shown that $\sup\|u^n\| < \infty$ and $L = \{x; |u(x)| = 1\} = x_0 H$ for some closed subgroup $H \subset G$. Again $L$ is a set of synthesis. If now $p \neq 2$, then closed subgroups $H \subset G$ are known only to have local spectral synthesis, i.e., $I_H \cap C_0 \subset J_H$; see Herz [8, p. 93].
(c) If $1 < p < \infty$, $G$ arbitrary and $u \in B^M_p(G)$ is such that $\sup_n ||u^n|| < \infty$, let $L = \{x; |u(x)| = 1\}$. Then $||u^n v||_{A_p} \to 0$ only for $v \in \text{Jac}_L$. It seems to be a hard, open question whether $L$ has (local) spectral synthesis in this case.\(^2\)

In the following, (i) improves the Choquet-Deny theorem [1, 13].

\textbf{Theorem 3.} Let $u \in B^M_p(G)$, $\lambda \in \mathbb{C}$ and $E_\lambda = \{\phi \in PM_p; u \cdot \phi = \lambda \phi\}$.

(i) dim $E_\lambda = n < \infty$ if and only if $u^{-1}\{\lambda\}$ is finite or void. In this case dim $E_\lambda = \text{card } u^{-1}\{\lambda\}$ and $E_\lambda = \{\Sigma a_i \delta_{a_i}; a_i \in u^{-1}\{\lambda\}, \alpha_i \in \mathbb{C}\}$. Note that $n = 0$ (i.e. $E_\lambda = \{0\}$) iff $u^{-1}\{\lambda\} = \emptyset$.

(ii) If $G$ is amenable and $(E_\lambda, || \cdot ||_{PM_p})$ is a reflexive Banach space, then $E_\lambda$ is finite dimensional.

(iii) If $G$ is discrete and contains the free group on two generators, then there exists $u \in B^M_2(G)$ for which $E_1 = \{\phi \in PM_2; u \cdot \phi = \phi\}$ is isomorphic to $l^2$ (a fortiori is reflexive infinite dimensional).

\textbf{Remark.} The reader should note that if $G = R = \hat{R}$ and $p = 2$, then $PM_2 = L^\infty(\hat{R}) = L^\infty(R)$ and every separable Banach space (reflexive or not) is isometric to a subspace of $PM_2$.

\textbf{Proof.} (i) If $v \in A^*_p, \phi \in PM_p = A^*_p$, then $\langle u \cdot \phi, v \rangle = \langle \phi, uv \rangle$. If $\phi \in E_\lambda, v \in A^*_p$, then $u \cdot (v \cdot \phi) = v \cdot (u \cdot \phi) = \lambda v \cdot \phi$. Thus $E_\lambda$ is an $A^*_p$-submodule of $PM_p$ which is $w^*$ closed. For any $a \in G, u \cdot \delta_a = u(a) \delta_a$. Thus $\{\delta_a; \delta_a \in E_\lambda\} = \{\delta_a; a \in u^{-1}\{\lambda\}\}$. The $\delta_a$'s are linearly independent; thus dim $E_\lambda \geq \text{card } u^{-1}\{\lambda\}$. Assume now that $u^{-1}\{\lambda\} = \{a_1, \ldots, a_n\}$ is finite. If $\Phi \in E_\lambda$, any $x \in \text{supp } \Phi$ is such that $\delta_x$ is a $w^*$ limit of a net $v_n \cdot \Phi$ with $v_n \in A^*_p$ (see [8, pp. 101, 118]). Hence $\delta_x \in E_\lambda$ and $x \in u^{-1}\{\lambda\} = \{a_1, \ldots, a_n\}$. A routine, well-known argument (see for example [7, proof of Theorem 1.3]) shows that $\Phi = \Sigma a_i \delta_{a_i}$ for some $a_i \in \mathbb{C}$. Thus dim $E_\lambda = \text{card } u^{-1}\{\lambda\}$ and $E_\lambda = \{\Sigma a_i \delta_{a_i}; a_i \in \mathbb{C}, a_i \in u^{-1}\{\lambda\}\}$ in this case. Note that $E_\lambda = \{0\}$ iff $u^{-1}\{\lambda\} = \emptyset$ is just the Tauberian condition (T) [8, p. 101].

(ii) If $G$ is amenable, any norm closed $A^*_p$-submodule of $PM_p$ which is reflexive is finite dimensional by our Theorem 1.3 in [7].

(iii) In this case $PM_2 \subset l^2(G)$ and $G$ contains an infinite Leinert set $L$, i.e., a set $L$ such that the subspace $N = \{\phi \in PM_2; \phi = 0 \text{ off } L\} = l^2(L)$ (as sets) and, for some $c > 0$, $||\phi||_2 \leq ||\phi||_{PM_2} \leq c||\phi||_2$ for all $\phi \in N$ (see [9, Satz 1]). A result of Figa-Talamanca and Picardello [3] implies that $1_L \in B^M_2(G)$; thus $N = \{\phi \in PM_2; 1_L \cdot \phi = \phi\}$. If $u = 1_L$ and $\lambda = 1$ then $E_1 = \{\phi \in PM_2; 1_L \cdot \phi = \phi\} = N$ is isomorphic to $l^2$.

H. P. Rosenthal proves in [18, p. 39] that if $G$ is abelian and $E \subset G$ closed, then $A_2/I_E$ is reflexive iff $E$ is finite.

We improve the result in [18] to all amenable groups $G$ and all $1 < p < \infty$. We also show that Rosenthal's result is false for $p = 2$ and discrete $G$ which contains some free nonabelian subgroup.

If $I \subset A^*_p$ is a closed subspace, $A^*_p/I$ is equipped with the quotient norm.

\(^2\)If the closed set $L$ is a coset of an amenable or normal subgroup $H$ (finite, compact, abelian or solvable are such), one still has that $\text{Jac}_L = I_L$ (see [8, pp. 92, 103] for more).
Theorem 4. Let \( I \subset A_p(G) \) be a closed ideal.

(a) If \( G \) is amenable, then \( A_p/I \) is reflexive if and only if it is finite dimensional. (Thus, if \( E \subset G \) is closed then \( A_p(E) = A_p/I_E \) is reflexive iff \( E \) is finite.)

(b) If \( G \) is discrete and contains the free group on two generators, then there is an infinite set \( E \subset G \) such that \( A_2(E) \) is isomorphic to \( l^2 \) (a fortiori is reflexive).

Proof. (a) Let \( N = (A_p/I)^* \). Then \( N = \{ \Phi \in PM_p(G); \langle \Phi, I \rangle = 0 \} \) and \( N \) is a \( \wedge^\ast \) closed \( A_p \)-submodule of \( PM_p \), since \( I \) is an ideal. Since \( G \) is amenable, we can apply our Theorem 1.3 of [7] to get that \( N \) (hence \( A_p/I \)) is finite dimensional. In case \( I = I_E \), \( \{ \delta_x; x \in E \} \) is a linearly independent subset of \( N \); hence \( E \) is finite.

(b) Let \( E \) be an infinite Leinert subset of \( G \). Then \( N = (A_2/I_E)^* \) is isomorphic as a Banach space to \( l^2(E) \) (see (iii) of the above theorem). Thus \( N^* = A_2/I_E \) also satisfies this condition.

H. P. Rosenthal proves in part of Theorem 2.12 [18, p. 53]) that if \( G \) is nondiscrete and abelian, then any nonzero ideal of \( A_2(G) \) contains a subspace isomorphic to \( l^1 \). We improve this theorem in

Theorem 5. (a) Let \( G \) be any nondiscrete locally compact group. Then every closed nonzero ideal \( I \) of \( A_p(G) \) contains a closed subspace isomorphic to \( l^1 \).

(b) If \( G \) is discrete infinite, then \( A_2(G) \) contains a closed ideal \( I \) isomorphic to \( l^2 \), a fortiori none of its closed subspaces is isomorphic to \( l^1 \) (due to M. A. Picardello [21]).

Remark. If \( G \) is compact abelian, \( A(G) = l^1(Z) \); hence \( l^1 \) cannot be replaced by any other infinite-dimensional Banach space nonisomorphic to \( l^1 \).

Proof. (a) Let \( Z = \{ x; v(x) = 0 \text{ for each } v \in I \} \). Then \( Z \neq G \) and \( Z \) is closed. Let \( a \in G \sim Z \) and \( V \) be a neighborhood of \( e \) such that \( aV^2 \cap Z = \emptyset \). Let \( V_n = V_n^{-1} \) be neighborhoods of \( e \) such that \( V_1 \subset V, V_n^2 \subset V_{n-1} \) if \( n \geq 2, m(V_n) \to 0 \). Let \( \Psi_n = m(V_n)^{-1}1_{V_n} \). Then, as is easily seen, \( \Psi_n \in A_p \cap C_0, \Psi_n(e) = 1 \) and \( \|\Psi_n\|_{A_p} \leq m(V_n)^{-1} \|1_{V_n}\|_{l^1} \|1_{V_n}\|_{\ell^1} = 1 \) \((1/p + 1/p' = 1)\). Thus \( \|\Psi_n\|_{A_p} = 1 = \Psi_n(e) \) and \( \Psi_n(x) = 0 \) if \( x \) is off \( V_n^2 \).

Let \( u_n = l_{a^{-1}} \Psi_n \), where \( l_{a^{-1}}u(x) = u(ax) \) for any \( u \in A_p, a, x \in G \). Then, by definition of the \( A_p \) norm [8, p. 97], \( \|u_n\|_{A_p} = 1 = u_n(a) \) and \( u_n(x) = 0 \) if \( x \) is off \( aV_n^2 \). Thus, if \( n \geq 2, u_n \in C_0 \cap A_p \) and \( u_n = 0 \) off \( aV_n^2 \); in particular off \( aV_1 \) and \( aV_1 \cap Z = \emptyset \). Thus \( u_n \) is in the smallest ideal whose zero set is \( Z \) and, in particular, in \( I \). We claim that no subsequence of \( \{u_n\} \) is weak Cauchy. In fact, assume that \( u_n \) is a weak Cauchy subsequence. If \( E = aV \), then \( u_n \in A_p(E) \) (\( v \in A_p; supp v \subset E \)). But \( A_p(E) \) is weakly sequentially complete by Lemma 18 of [20]. Hence \( u_n \to u \sigma(A_p, PM_p) \) for some \( u \in A_p \). In particular, for each \( \mu \in M(G) \), \( \int u_n \mu \to \int u \mu \). By taking \( \mu = \delta_x \), we get \( u(a) = 1 \). And if \( x \notin aV_n^2 \), then \( u_n(x) = 0 \) if \( n \geq k \). Hence \( u(x) = 0 \) if \( x \notin \cap_n aV_n^2 \). Now \( m(V_n^2) \leq m(V_{n-1}) \to 0 \). Hence \( \cap_n aV_n^2 \) has void interior. But \( u \in A_p \subset C_0(G) \); hence \( \{ a \} \subset \{ x; |u(x)| > \frac{1}{2} \} \subset \cap_n aV_n^2 \). This is a contradiction. It follows that no subsequence of \( u_n \) is weak Cauchy. We now apply H. P. Rosenthal’s deep Theorem 1 of [19, p. 805] and get that some subsequence \( u_n \), of \( u_n \) is isomorphic to a canonical \( l^1 \) basis.
(b) We follow the notation of Picardello [21]. By Theorem 1 of [21] every infinite subset of $G$ contains a subset $E$ which is a $\Lambda(4)$ set. By Proposition 2 of [21] and the remark after it, $E$ is also a $\Lambda(2)$ set. However, by Remark 4 (after Definition 5 of [21]), $L^1(\Gamma)$ is isometrically isomorphic to $A_2(G)$ [1^2]. It follows that the ideal $I = \{ u \in A_2(G) ; u = 0 \text{ off } E \}$ with $A_2(G)$-norm is isomorphic to $I^2$.

The following theorem is related to the main result of J. E. Gilbert [5] on existence of projections which commute with convolution, onto $w^*$ closed $A(G)$ submodules of $PM_2(G)$.

Let $S \subset B_p^M(G)$ be a norm bounded semigroup (with respect to multiplication). For example, $S = \{ u^n ; n \geq 1 \}$, where $u \in B_p^M$ satisfies $\sup \| u^n \| < \infty$, is such a semigroup. Theorems 6.2 and 6.20 of Schreiber [15] clarify to some extent the spectrum of submodules $F$ which can be expressed as in the next theorem.

**Theorem 6.** Let $S \subset B_p^M(G)$ be a norm bounded semigroup, and $F = \{ \phi \in PM_p ; u \cdot \phi = \phi \text{ for each } u \in S \}$. Then there exists a bounded linear onto projection $P : PM_p \to F$ such that $P(v \cdot \phi) = v \cdot P\phi$ for all $v \in A_p$.

**Proof.** For each $\Phi \in PM_p$ let $K_\Phi = w^* \text{cl}(CoS \cdot \phi)$, where $S \cdot \Phi = \{ u \cdot \Phi ; u \in S \}$ and $Co$ denotes convex hull. Each $K_\Phi$ is a $w^*$ compact convex set which satisfies $s \cdot K_\Phi \subset K_\Phi$ for each $s \in S$. Furthermore, each operator $\psi \to s \cdot \psi$ on $PM_p$ is $w^*$-$w^*$ continuous, and the semigroup of operators $S$ on $PM_p$ is commutative. Hence, by the Markov-Kakutani theorem, $K_\Phi \cap F \neq \emptyset$ for each $\Phi$ in $PM_p$. We note now that $F$ is a $w^*$ closed $A_p$-submodule of $PM_p$, and that the $w^*$ operator closure of $CoS$ in the space $L(PM_p)$ of operators from $PM_p$ to $PM_p$ (denote this set by $Co^*S$) is a semigroup which is a $w^*$-ot compact set; see A. T. Lau [11] just preceding Theorem 2.1. (Here $w^*$-ot denotes the $w^*$ operator topology on $L(PM_p)$.)

We apply now Theorem 2.1 of A. T. Lau (and the remark after its proof) [11] with $X = PM_p$ and get that there exists an operator $P \in Co^*(S)$ which is $F$-stationary on $X = PM_p$, i.e., such that $P\Phi \in F$ for each $\Phi$ in $PM_p$. Note here that $S$ need not consist of only isometric operators on $PM_p$ (as stated in the introduction of [11]). Lau’s proof works for any norm bounded semigroup. Let $u_a \in CoS$ be such that $(u \cdot \Phi, v) \to \langle P\Phi, v \rangle$ for each $\Phi \in PM_p$ and $v \in A_p$. Let $Q : PM_p \to PM_p$ be $w^*$-$w^*$ continuous and commute with each $u \in S$, i.e., $Q(u \cdot \phi) = u \cdot Q\phi$ for each $\phi \in PM_p$. Then this holds also for each $u \in CoS$. But then $(u \cdot Q\phi, v) = \langle Q(u_a \cdot \phi), v \rangle \to \langle Q(\Phi), v \rangle$, and the left side converges to $(Q(P\Phi), v)$ for all $v \in A_p$. Hence, $P$ commutes with every $w^*$ continuous operator $Q : PM_p \to PM_p$ which commutes with each operator $\phi \to s \cdot \phi$ for each $s \in S$. For any $v \in A_p$ the operator $Q_s(\phi) = v \cdot \phi$ is such an operator. It follows that $P(v \cdot \phi) = v \cdot P\Phi$ for all $v \in A_p$ and all $\Phi \in PM_p$. If now $\Phi \in F$, then $u \cdot \Phi = \Phi$ for each $u \in CoS$. Thus $P\Phi = \Phi$ since $P \in Co^*(S)$. But $P(PM_p) \subset F$, since $P$ is $F$-stationary. It follows that $P$ is the required projection onto $F$.

**Remark.** (a) Let $P$ denote the set of all $F$-stationary operators $P \in Co^*(S)$ on $PM_p$. Then Lau’s Theorem 2.1 [11] implies that $\{ (Co^*S)(\phi) \} \cap F = \{ P\phi ; P \in P \}$ for each $\phi \in PM_p$.

(b) The main idea in the above proof is due to Anthony Lau and is also used in Theorem 2 of [17].
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