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ON SOME PROPERTIES OF THE BANACH ALGEBRAS $A_p(G)$ FOR LOCALLY COMPACT GROUPS

Dedicated to my teacher Rafael Artzy with gratitude and respect

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ABSTRACT. We strengthen and improve theorems of Choquet-Deny and of Foguel concerning convolution equations and iterates of convolution of a measure to all algebras $A_p(G)$ and all locally compact groups. Furthermore, we improve results of H. P. Rosenthal on ideals of $A(G)$ to the algebras $A_p(G)$ and show that some hold for amenable groups but not for free nonabelian groups. Finally, we improve a (possibly) weak version of a theorem of Gilbert on projections onto some subspaces of $L^\infty(G)$ to all locally compact groups.

Introduction. In [**13**] T. Ramsey and Y. Weit provide new proofs for the following theorem of S. Foguel and of Choquet-Deny concerning iterates of convolutions of a measure on a locally compact abelian group G with dual Γ.[1]

THEOREM (FOGUEL). *Let G be an l.c.a. group and $\mu \in M(G)$ be such that* $\sup_n\|\mu^n\| < \infty$. *Then* $\lim\|\mu^n * f\|_1 = 0$ *for each* $f \in I_e = \{f \in L^1(G);\ \hat{f}(e) = 0\}$ *if and only if* $|\mu(\gamma)| < 1$ *for all* $\gamma \in \Gamma \sim \{e\}$.

Here μ^n is the n-times convolution power of μ and e denotes the unit of Γ.

THEOREM (CHOQUET-DENY). *Let G be an l.c.a. group and $\mu \in M(G)$. The following are equivalent*:

(i) *for* $f \in L^\infty(G)$, $\mu * f = f$ *implies* $f =$ *constant.*

(ii) $\mu(\gamma) \neq 1$ *for* $\gamma \in \Gamma \sim \{e\}$.

We strengthen and improve (the dual version of) both results to all locally compact groups and all algebras $A_p(G)$, $B_p^M(G)$, $PM_p(G)$ in the first two theorems of the paper and in the remarks after them. (If G is abelian and $p = 2$, then $A_2(G) = A(G) = L^1(\Gamma)\hat{}$, $B_2^M(G) = B(G) = M(\Gamma)\hat{}$ and $PM_2(G) = L^\infty(\Gamma)$.) Furthermore, we point out that if, for some $\lambda \in \mathbf{C}$ and $u \in B_p^M$, $E_\lambda = \{\phi \in PM_p;\ u \cdot \phi = \lambda\phi\}$ is a reflexive Banach space and if G is amenable, then E_λ is finite dimensional. This is false if G is discrete and contains the free group on two generators (via existence of Leinert sets in G).

In Theorem 4 we improve a result of H. P. Rosenthal [**18**, p. 39] to all amenable groups G. We show that, if for some closed ideal I, A_p/I is a reflexive Banach space and if G is amenable, then A_p/I is finite dimensional. Again, using Leinert sets, we

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[1] Thanks are due to T. Ramsey for providing us with a preprint of [**13**] and for discussions related to it.

get a counterexample in case G is discrete and contains a free group on two generators.

In Theorem 5 we improve the following result of H. P. Rosenthal, which is part of Theorem 2.12 on p. 53 in **[18]**: *If G is any nondiscrete locally compact abelian group, then any nonzero ideal of $A(G)$ contains an isomorphic copy of l^1.* We show that this result is true for all nondiscrete G and for all algebras $A_p(G)$. This is false if G is discrete and $p = 2$, as shown by M. A. Picardello **[21]**.

In Theorem 6 we prove a result related to the beautiful main theorem of J. E. Gilbert on existence of projections onto w* closed translation invariant subspaces of $L^\infty(G)$. Again this is done in the framework of the algebras A_p, B_p^M, PM_p.

The reader not familiar with the algebras A_p may find the results that follow of interest even for $p = 2$ and abelian G.

Definitions and notation. $\mathbf{C}$ will denote the complex field. G will always denote a locally compact group, $C_0(G)$ ($C_{00}(G)$) the continuous functions on G, which tend to 0 at ∞ (with compact support). $L^p(G)$, $1 \leqslant p \leqslant \infty$, will be the usual spaces of p-integrable functions with respect to a fixed left Haar measure m and $\|f\|_p = (\int |f|^p\, dm)^{1/p}$, $\|f\|_\infty = \operatorname{ess\,sup}|f(x)|$. We follow Herz **[8]** for notation and properties of the Banach algebras $A_p(G) = A_p$. One has that $A_2(G) = A(G)$ is the Fourier algebra of G à la Eymard **[2]**. We denote by $\|v\|_{A_p}$ the norm in A_p (or just $\|v\|$ when the context is clear). We denote by $B_p^M = B_p^M(G)$ the set of bounded complex functions u on G such that $uv \in A_p(G)$ for all $v \in A_p$. The norm in B_p^M is given by $\|u\|_M = \sup\{\|uv\|_{A_p};\ \|v\|_{A_p} = 1\}$. If G is abelian with dual Γ, then $A_2(G) = A(G) = L^1(\Gamma)\hat{}$ and $B_2^M(G) = B(G) = M(\Gamma)\hat{}$, where $M(\Gamma)$ is the Banach algebra of bounded complex measures on Γ and $\hat{}$ denotes Fourier transform. $PM_p(G)$ is the Banach space dual of A_p as in **[8]**. If G is abelian, then $PM_2(G) = L^\infty(\Gamma)$. We define the module action of B_p^M on PM_p by $\langle u \cdot \phi, v\rangle = \langle \phi, uv\rangle$ for $\phi \in PM_p$, $v \in A_p$, $u \in B_p^M$. If $v \in A_p$, then $\operatorname{supp} v$ denotes the closure in G of $\{x;\ v(x) \neq 0\}$.

If C is a subset of A_p, then $\overline{C}$ will denote the norm closure of C in A_p.

Some interesting properties of the algebras $A_p(G)$ for abelian G have been obtained by N. Lohoue in C. R. Acad. Sci. Paris Ser. A **273** (1971), 893–896.

If X is a Banach space, then $L(X)$ will denote the bounded linear operators T: $X \to X$ with $\|T\| = \sup\{\|Tx\|;\ \|x\| = 1\}$. If A, B are subsets of C, then $A \sim B$ will denote the set-theoretical difference of A and B. And if τ is a topology on C, then $\tau \operatorname{cl} A$ will denote the τ-closure of A in C.

The following lemma was obtained independently of the proofs given in **[13]**.

LEMMA 1. *Let $u \in B_p^M(G)$ be such that $|u(x)| \leqslant 1$ for all x and, let $L = \{x;\ |u(x)| = 1\}$. If $v \in C_{00} \cap A_p(G)$ is such that $\{\operatorname{supp} v\} \cap L = \varnothing$, then $\|u^n v\|_{A_p} \to 0$ as $n \to \infty$. ($L = \varnothing$ is allowed.)*

PROOF. Let $S = \operatorname{supp} v$. S is compact and there exists a symmetric neighborhood of e, V such that $SV^2 \cap LV = \varnothing$ and $\overline{V}$ is compact. For x in G define

$$g(x) = \lambda(V)^{-1}[1_{SV} * 1_V](x) = \lambda(V)^{-1}\lambda(xV \cap SV).$$

Then $g \in A_p$ by the definition of A_p, $g(x) = 1$ on S, $g(x) = 0$ if x is off SV^2 and $0 \leqslant g(x) \leqslant 1$ for all x. Furthermore, $ug \in A_p \cap C_{00}$. Hence $|u(x)g(x)| < 1$ for all x

in G (if $x \in L$ then $g(x) = 0$, and if $x \notin L$ then $|u(x)| < 1$, while $0 \leqslant g \leqslant 1$). Let $d = \sup\{|u(x)g(x)|;\ x \in G\} < 1$. The maximal ideal space of $A_p(G)$ is G [**8**, p. 102], hence the spectral radius of ug is $d = \lim_n \|(ug)^n\|^{1/n} < 1$. Now choose $\delta > 0$ such that $d + \delta < 1$. Then for some n_0 we have $\|(ug)^n\|_{A_p} < (d+\delta)^n$ if $n \geqslant n_0$. Hence $\|(ug)^n\|_{A_p} \to 0$. But $g^n(x) = 1$ if $x \in S$; thus $g^n v = v$. It follows that

$$\|u^n v\|_{A_p} = \|u^n g^n v\|_{A_p} \leqslant \|u^n g^n\|_{A_p} \|v\|_{A_p} \to 0.$$

Remark 1. Note that *no* assumption on the boundedness of $\|u^n\|_M$ is made in the above lemma. If, however, $\sup\|u^n\| = C < \infty$, then $|u^n(x)| \leqslant C$ for all n; thus $|u(x)| \leqslant 1$ for all x.

Let $L \subset G$ be closed. Let $J_L = \{v \in A_p \cap C_{00};\ \operatorname{supp} v \cap L = \varnothing\}$ and $I_L = \{v \in A_p;\ v = 0 \text{ on } L\}$. Clearly $\bar{J}_L \subset I_L$.

Remark 2. If $u \in B_p^M$ is such that $\|u^n v\| \to 0$ for each $v \in J_L$ (where $L \subset G$ is any closed set), then $|u(x)| < 1$ for each $x \in G \sim L$. Since if $|u(x_0)| = 1$, $x_0 \in G \sim L$, then there is some $v \in J_L$ such that $v(x_0) = 1$. Then $1 = |u^n(x_0)v(x_0)| \leqslant \|u^n v\|$.

Theorem 2. *Let* $u \in B_p^M(G)$ *be such that* $\sup\|u^n\|_M = C < \infty$, *and let* $L = \{x;\ |u(x)| = 1\}$. *Then* $\|u^n v\|_{A_p} \to 0$ *for each* $v \in \bar{J}_L$.

(*) *If L is a set of spectral synthesis then* $\|u^n v\|_{A_p} \to 0$ *for each* $v \in I_L$.

Proof. Let $v \in \bar{J}_L$, $\varepsilon > 0$. Let $v_0 \in J_L$ be such that $\|v - v_0\| < \varepsilon$. then

$$\|u^n v\| \leqslant \|u^n(v - v_0)\| + \|u^n v_0\| \leqslant C\varepsilon + \|u^n v_0\| \to C\varepsilon$$

by the above lemma. If L has spectral synthesis, then $\bar{J}_L = I_L$.

Remarks. In many cases the condition $\sup\|u^n\|_M < \infty$ forces the set L to be a set of spectral synthesis:

(a) Let $p = 2$ and G *abelian*. Assume that $u \in B(G) = M(\hat{G})\,\hat{}$ is such that $\sup\|u^n\|_{B(G)} < \infty$. Then $\sup\{|u(x)|;\ x \in G\} \leqslant 1$ and $L = \{x;\ |u(x)| = 1\}$ is a closed subset of the coset ring of G and, as such, is even a strong Ditkin set, by J. E. Gilbert [**5, 6**] or B. Schreiber [**15**, Theorem 6.2 and **14**, Theorem 2.6]. A fortiori, L is a set of spectral synthesis.

Foguel's result is thus improved in the

Corollary. *Let G be l.c.a.,* $\mu \in M(G)$ *satisfy* $\sup\|\mu^n\| < \infty$, *and let* $L = \{\gamma \in \Gamma;\ |\hat{\mu}(\gamma)| = 1\}$. *Then* $\|\mu^n * f\|_1 \to 0$ *iff* $f \in I_L = \{f \in L^1(G);\ \hat{f} = 0 \text{ on } L\}$. (*Since clearly* $|\hat{f}(r)| \leqslant \|\mu^n * f\|_1$ *if* $r \in L$.)

(b) Let $p = 2$, G arbitrary. Let $B(G)$ be as in [**2**], and let $u \in B(G) \subset B_2^M(G)$ be a positive definite function such that $u(e) = 1$. Then $\|u\|_{B(G)} = 1$ and $L = \{x;\ |u(x)| = 1\}$ is a closed (not necessarily normal) subgroup of G. Then $\|u^n\|_M = \|u^n\|_{B(G)} = 1$ and L has spectral synthesis, as shown by Takesake and Tatsuma in [**16**]. If $u \in B(G)$ only satisfies $|u(x_0)| = \|u\| = 1$ at some x_0 in G, then it can easily be shown that $\sup\|u^n\| < \infty$ and $L = \{x;\ |u(x)| = 1\} = x_0 H$ for some closed subgroup $H \subset G$. Again L is a set of synthesis. If now $p \neq 2$, then closed subgroups $H \subset G$ are known only to have local spectral synthesis, i.e., $I_H \cap C_{00} \subset \bar{J}_H$; see Herz [**8**, p. 93].

(c) If $1 < p < \infty$, G arbitrary and $u \in B_p^M(G)$ is such that $\sup_n \|u^n\| < \infty$, let $L = \{x; |u(x)| = 1\}$. Then $\|u^n v\|_{A_p} \to 0$ only for $v \in \bar{J}_L$. It seems to be a hard, open question whether L has (local) spectral synthesis in this case.[2]

In the following, (i) improves the Choquet-Deny theorem [**1**, **13**].

THEOREM 3. *Let* $u \in B_p^M(G)$, $\lambda \in \mathbf{C}$ *and* $E_\lambda = \{\phi \in PM_p;\ u \cdot \phi = \lambda\phi\}$.

(i) $\dim E_\lambda = n < \infty$ *if and only if* $u^{-1}\{\lambda\}$ *is finite or void. In this case* $\dim E_\lambda = \operatorname{card} u^{-1}\{\lambda\}$ *and* $E_\lambda = \{\Sigma\alpha_i\delta_{a_i};\ a_i \in u^{-1}\{\lambda\},\ \alpha_i \in \mathbf{C}\}$. *Note that* $n = 0$ (*i.e.* $E_\lambda = \{0\}$) *iff* $u^{-1}\{\lambda\} = \varnothing$.

(ii) *If G is amenable and* $(E_\lambda, \|\ \|_{PM_p})$ *is a reflexive Banach space, then* E_λ *is finite dimensional.*

(iii) *If G is discrete and contains the free group on two generators, then there exists* $u \in B_2^M(G)$ *for which* $E_1 = \{\phi \in PM_2;\ u \cdot \phi = \phi\}$ *is isomorphic to* l^2 (*a fortiori is reflexive infinite dimensional*).

REMARK. The reader should note that if $G = R = \hat{R}$ and $p = 2$, then $PM_2 = L^\infty(\hat{R}) = L^\infty(R)$ and every separable Banach space (reflexive or not) is isometric to a subspace of PM_2.

PROOF. (i) If $v \in A_p$, $\phi \in PM_p = A_p^*$, then $\langle u \cdot \phi, v\rangle = \langle \phi, uv\rangle$. If $\phi \in E_\lambda$, $v \in A_p$, then $u \cdot (v \cdot \phi) = v \cdot (u \cdot \phi) = \lambda v \cdot \phi$. Thus E_λ is an A_p-submodule of PM_p which is w* closed. For any $a \in G$, $u \cdot \delta_a = u(a)\delta_a$. Thus $\{\delta_x;\ \delta_x \in E_\lambda\} = \{\delta_x;\ x \in u^{-1}\{\lambda\}\}$. The δ_x's are linearly independent; thus $\dim E_\lambda \geqslant \operatorname{card} u^{-1}\{\lambda\}$. Assume now that $u^{-1}\{\lambda\} = \{a_1, \ldots, a_n\}$ is finite. If $\Phi \in E_\lambda$, any $x \in \operatorname{supp} \Phi$ is such that δ_x is a w* limit of a net $v_\alpha \cdot \Phi$ with $v_\alpha \in A_p$ (see [**8**, pp. 101, 118]). Hence $\delta_x \in E_\lambda$ and $x \in u^{-1}\{\lambda\} = \{a_1, \ldots, a_n\}$. Thus $\operatorname{supp} \Phi \subset \{a_1, \ldots, a_n\}$. A routine, well-known argument (see for example [7, proof of Theorem 1.3]) shows that $\Phi = \sum_1^n \alpha_i \delta_{a_i}$ for some $\alpha_i \in \mathbf{C}$. Thus $\dim E_\lambda = \operatorname{card} u^{-1}\{\lambda\}$ and $E_\lambda = \{\Sigma\alpha_i\delta_{a_i};\ \alpha_i \in \mathbf{C},\ a_i \in u^{-1}\{\lambda\}\}$ in this case. Note that $E_\lambda = \{0\}$ iff $u^{-1}\{\lambda\} = \varnothing$ is just the Tauberian condition (T) [**8**, p. 101].

(ii) If G is amenable, any norm closed A_p-submodule of PM_p which is reflexive is finite dimensional by our Theorem 1.3 in [**7**].

(iii) In this case $PM_2 \subset l^2(G)$ and G contains an infinite Leinert set L, i.e., a set L such that the subspace $N = \{\phi \in PM_2,\ \phi = 0 \text{ off } L\} = l^2(L)$ (as sets) and, for some $c > 0$, $\|\phi\|_{l^2} \leqslant \|\phi\|_{PM_2} \leqslant c\|\phi\|_{l^2}$ for all $\phi \in N$ (see [**9**, Satz 1]). A result of Figa-Talamanca and Picardello [3] implies that $1_L \in B_M^2(G)$; thus $N = \{\phi \in PM_2;\ 1_L \cdot \phi = \phi\}$. If $u = 1_L$ and $\lambda = 1$ then $E_1 = \{\phi \in PM_2;\ 1_L \cdot \phi = \phi\} = N$ is isomorphic to l^2.

H. P. Rosenthal proves in [**18**, p. 39] that if G is abelian and $E \subset G$ closed, then A_2/I_E is reflexive iff E is finite.

We improve the result in [**18**] to all *amenable* groups G and all $1 < p < \infty$. We also show that Rosenthal's result is false for $p = 2$ and discrete G which contains some free nonabelian subgroup.

If $I \subset A_p$ is a closed subspace, A_p/I is equipped with the quotient norm.

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[2] If the closed set L is a coset of an amenable or normal subgroup H (finite, compact, abelian or solvable are such), one still has that $\bar{J}_L = I_L$ (see [**8**, pp. 92, 103] for more).

THEOREM 4. *Let $I \subset A_p(G)$ be a closed ideal.*

(a) *If G is amenable, then A_p/I is reflexive if and only if it is finite dimensional. (Thus, if $E \subset G$ is closed then $A_p(E) = A_p/I_E$ is reflexive iff E is finite.)*

(b) *If G is discrete and contains the free group on two generators, then there is an infinite set $E \subset G$ such that $A_2(E)$ is isomorphic to l^2 (a fortiori is reflexive).*

PROOF. (a) Let $N = (A_p/I)^*$. Then $N = \{\Phi \in PM_p(G); \langle \Phi, I \rangle = 0\}$ and N is a w* closed A_p-submodule of PM_p, since I is an ideal. Moreover, N is also reflexive. Since G is amenable, we can apply our Theorem 1.3 of [**7**] to get that N (hence A_p/I) is finite dimensional. In case $I = I_E$, $\{\delta_x: x \in E\}$ is a linearly independent subset of N; hence E is finite.

(b) Let E be an infinite Leinert subset of G. Then $N = (A_2/I_E)^*$ is isomorphic as a Banach space to $l^2(E)$ (see (iii) of the above theorem). Thus $N^* = A_2/I_E$ also satisfies this condition.

H. P. Rosenthal proves in part of Theorem 2.12 [**18**, p. 53]) that if G is nondiscrete and *abelian*, then any nonzero ideal of $A_2(G)$ contains a subspace isomorphic to l^1. We improve this theorem in

THEOREM 5. (a) *Let G be any nondiscrete locally compact group. Then every closed nonzero ideal I of $A_p(G)$ contains a closed subspace isomorphic to l^1.*

(b) *If G is discrete infinite, then $A_2(G)$ contains a closed ideal I isomorphic to l^2, a fortiori none of its closed subspaces is isomorphic to l^1 (due to M. A. Picardello* [**21**]).

REMARK. If G is compact abelian, $A(G) = l^1(Z)$; hence l^1 cannot be replaced by any other infinite-dimensional Banach space nonisomorphic to l^1.

PROOF. (a) Let $Z = \{x;\ v(x) = 0 \text{ for each } v \in I\}$. Then $Z \neq G$ and Z is closed. Let $a \in G \sim Z$ and V be a neighborhood of e such that $aV^2 \cap Z = \varnothing$. Let $V_n = V_n^{-1}$ be neighborhoods of e such that $\overline{V}_1 \subset V$, $V_n^2 \subset V_{n-1}$ if $n \geq 2$, $m(V_n) \to 0$. Let $\Psi_n = m(V_n)^{-1} 1_{V_n} * 1_{V_n}$. Then, as is easily seen, $\Psi_n \in A_p \cap C_{00}$, $\Psi_n(e) = 1$ and $\|\Psi_n\|_{A_p} \leq m(V_n)^{-1} \|1_{V_n}\|_p \|1_{V_n}\|_{p'} = 1$ $(1/p + 1/p' = 1)$. Thus $\|\Psi_n\|_{A_p} = 1 = \Psi_n(e)$ and $\Psi_n(x) = 0$ if x is off V_n^2.

Let $u_n = l_{a^{-1}} \Psi_n$, where $l_a u(x) = u(ax)$ for any $u \in A_p$, a, $x \in G$. Then, by definition of the A_p norm [**8**, p. 97], $\|u_n\|_{A_p} = 1 = u_n(a)$ and $u_n(x) = 0$ if x is off aV_n^2. Thus, if $n \geq 2$, $u_n \in C_{00} \cap A_p$ and $u_n = 0$ off aV_2^2, in particular off $a\overline{V}_1$ and $a\overline{V}_1 \cap Z = \varnothing$. Thus u_n is in the smallest ideal whose zero set is Z and, in particular, in I. We claim that no subsequence of $\{u_n\}$ is weak Cauchy. In fact, assume that u_{n_i} is a weak Cauchy subsequence. If $E = a\overline{V}$, then $u_{n_i} \in A_E^p(G) = \{v \in A_p;\ \operatorname{supp} v \subset E\}$. But $A_E^p(G)$ is weakly sequentially complete by Lemma 18 of [**20**]. Hence $u_{n_i} \to u$ $\sigma(A_p, PM_p)$ for some $u \in A_p^E$. In particular, for each $\mu \in M(G)$, $\int u_{n_i} \mu \to \int u \, d\mu$. By taking $\mu = \delta_a$, we get $u(a) = 1$. And if $x \notin aV_k^2$, then $u_{n_i}(x) = 0$ if $n_i \geq k$. Hence $u(x) = 0$ if $x \notin \bigcap_n aV_n^2$. Now $m(V_n^2) \leq m(V_{n-1}) \to 0$. Hence $\bigcap_1^\infty aV_n^2$ has void interior. But $u \in A_p \subset C_0(G)$; hence $\{a\} \subset \{x;\ |u(x)| > \frac{1}{2}\} \subset \bigcap_n aV_n^2$. This is a contradiction. It follows that no subsequence of u_n is weak Cauchy. We now apply H. P. Rosenthal's deep Theorem 1 of [**19**, p. 805] and get that some subsequence u_{n_i} of u_n is isomorphic to a canonical l^1 basis.

(b) We follow the notation of Picardello [**21**]. By Theorem 1 of [**21**] every infinite subset of G contains a subset E which is a $\Lambda(4)$ set. By Proposition 2 of [**21**] and the remark after it, E is also a $\Lambda(2)$ set. However, by Remark 4 (after Definition 5 of [**21**]), $L^1(\Gamma)$ $[L^2(\Gamma)]$ is isometrically isomorphic to $A_2(G)$ $[l^2]$. It follows that the ideal $I = \{u \in A_2(G);\ u = 0 \text{ off } E\}$ with $A_2(G)$-norm is isomorphic to l^2.

The following theorem is related to the main result of J. E. Gilbert [**5**] on existence of projections which commute with convolution, onto w* closed $A(G)$ submodules of $PM_2(G)$.

Let $S \subset B_p^M(G)$ be a norm bounded semigroup (with respect to multiplication). For example, $S = \{u^n;\ n \geqslant 1\}$, where $u \in B_P^M$ satisfies $\sup\|u^n\| < \infty$, is such a semigroup. Theorems 6.2 and 6.20 of Schreiber [**15**] clarify to some extent the spectrum of submodules F which can be expressed as in the next theorem.

THEOREM 6. *Let $S \subset B_P^M(G)$ be a norm bounded semigroup, and $F = \{\phi \in PM_p;$ $u \cdot \phi = \phi$ for each u in $S\}$. Then there exists a bounded linear onto projection P: $PM_p \to F$ such that $P(v \cdot \phi) = v \cdot P\phi$ for all v in A_p.*

PROOF. For each $\Phi \in PM_p$ let $K_\Phi = \text{w*}\,\text{cl}\{\text{Co}\, S \cdot \phi\}$, where $S \cdot \Phi = \{u \cdot \Phi;$ $u \in S\}$ and Co denotes convex hull. Each K_Φ is a w* compact convex set which satisfies $s \cdot K_\Phi \subset K_\Phi$ for each $s \in S$. Furthermore, each operator $\psi \to s \cdot \psi$ on PM_p is w*-w* continuous, and the semigroup of operators S on PM_p is commutative. Hence, by the Markov-Kakutani theorem, $K_\Phi \cap F \neq \varnothing$ for each Φ in PM_p. We note now that F is a w* closed A_p-submodule of PM_p, and that the w* operator closure of $\text{Co}\, S$ in the space $L(PM_p)$ of operators from PM_p to PM_p (denote this set by $\overline{\text{Co}}^*\, S$) is a semigroup which is a w* ot compact set; see A. T. Lau [**11**] just preceding Theorem 2.1. (Here w* ot denotes the w* operator topology on $L(PM_p)$.)

We apply now Theorem 2.1 of A. T. Lau (and the remark after its proof) [**11**] with $X = PM_p$ and get that there exists an operator $P \in \overline{\text{Co}}^*(S)$ which is F-stationary on $X = PM_P$, i.e., such that $P\Phi \in F$ for each Φ in PM_p. Note here that S need not consist of only isometric operators on PM_p (as stated in the introduction of [**11**]). Lau's proof works for any norm bounded semigroup. Let $u_\alpha \in \text{Co}\, S$ be such that $\langle u_\alpha \cdot \phi, v\rangle \to \langle P\phi, v\rangle$ for each $\phi \in PM_p$ and $v \in A_p$. Let Q: $PM_p \to PM_p$ be w*-w* continuous and commute with each $u \in S$, i.e., $Q(u \cdot \phi) = u \cdot Q\phi$ for each $\phi \in PM_p$. Then this holds also for each $u \in \text{Co}\, S$. But then $\langle u_\alpha \cdot Q\phi, v\rangle =$ $\langle Q(u_\alpha \phi), v\rangle \to \langle Q(P\phi), v\rangle$, and the left side converges to $\langle P(Q\phi), v\rangle$ for all $v \in A_p$. Hence, P commutes with every w* continuous operator Q: $PM_p \to PM_p$ which commutes with each operator $\phi \to s \cdot \phi$ for each $s \in S$. But for any $v \in A_p$ the operator $Q_v(\phi) = v \cdot \phi$ is such an operator. It follows that $P(v \cdot \phi) = v \cdot P\Phi$ for all $v \in A_p$ and all $\Phi \in PM_p$. If now $\Phi \in F$, then $u \cdot \Phi = \Phi$ for each $u \in \text{Co}\, S$. Thus $P\Phi = \Phi$ since $P \in \overline{\text{Co}}^*(S)$. But $P(PM_p) \subset F$, since P is F-stationary. It follows that P is the required projection onto F.

REMARK. (a) Let $\mathscr{P}$ denote the set of all F-stationary operators $P \in \overline{\text{Co}}^*(S)$ on PM_p. Then Lau's Theorem 2.1 [**11**] implies that $\{(\overline{\text{Co}}^*\, S)(\phi)\} \cap F = \{P\phi;\ P \in \mathscr{P}\}$ for each $\phi \in PM_p$.

(b) The main idea in the above proof is due to Anthony Lau and is also used in Theorem 2 of [**17**].

References

1. G. Choquet and J. Deny, *Sur l'équation de convolution* $\mu = \mu * \sigma$, C. R. Acad. Sci. Paris **250** (1960), 799–801.

2. P. Eymard, *L'algebre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236.

3. A. Figa-Talamanca and M. Picardello, *Multiplicateurs de $A(G)$ qui ne sont pas dans $B(G)$*, C. R. Acad. Sci. Paris Ser. A **227** (1973), 117–119.

4. S. R. Foguel, *On iterates of convolutions*, Proc. Amer. Math. Soc. **47** (1975), 368–370.

5. J. E. Gilbert, *On projections of $L^\infty(G)$ onto translation-invariant subspaces*, Proc. London Math. Soc. **19** (1969), 69–88.

6. ______, *On a strong form of spectral synthesis*, Ark. Mat. **7** (1969), 571–575.

7. E. E. Granirer, *Some results on $A_p(G)$ submodules of $PM_p(G)$*, Colloq. Math. (to appear).

8. C. Herz, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier (Grenoble) **23** (1973), 91–123.

9. M. Leinert, *Multiplication gewisser discreter Gruppen*, Studia Math. **52** (1974), 149–158.

10. Francoise Lust, *L'espace des fonctions presque-periodiques dont le spectre est contenu dans un ensemble compact denombrable a la propriété de Schur*, Colloq. Math. **41** (1979), 273–284.

11. Anthony To-Ming Lau, *Semigroups of operators on dual Banach spaces*, Proc. Amer. Math. Soc. **54** (1976), 393–396.

12. S. Massaheb, *A note on Fourier transforms*, Bull. London Math. Soc. **11** (1979).

13. T. Ramsey and Yitzhak Weit, *Ergodic and mixing properties of measures on locally compact abelian groups*, Proc. Amer. Math. Soc. **92** (1984), 519–520.

14. B. M. Schreiber, *On the coset ring and strong Ditkin set*, Pacific J. Math. **32** (1970), 805–812.

15. ______, *Measures with bounded convolution powers*, Trans. Amer. Math. Soc. **151** (1970), 405–431.

16. M. Takesaki and N. Tatsuma, *Duality and subgroups*. II, J. Funct. Anal. **11** (1972), 184–190.

17. A. Lau and V. Losert, *W^*-closed complemented invariant subspaces of $L^\infty(G)$ and amenable locally compact groups* (submitted).

18. H. P. Rosenthal, *Projections onto translation invariant subspaces of $L^p(G)$*, Mem. Amer. Math. Soc. No. 63 (1966).

19. ______, *Some recent discoveries in the isomorphic theory of Banach spaces*, Bull. Amer. Math. Soc. **84** (1978), 803–831.

20. E. E. Granirer, *On some spaces of linear functionals on the algebras $A_p(G)$, for locally compact groups*, Colloq. Math. (to appear).

21. M. A. Picardello, *Lacunary sets in discrete noncommutative groups*, Boll. Un. Mat. Ital. **4** (1973), 494–508.

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