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MONOTONE L_1-APPROXIMATION ON THE UNIT n-CUBE

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ABSTRACT. Let Ω be the unit n-cube $[0,1]^n$, and let M be the set of all real-valued functions on Ω, each of which is nondecreasing in each variable separately. If f: $\Omega \to \mathbb{R}$ is continuous, we show that there exists an (essentially) unique, best L_1-approximation, f_1, to f by elements of M, and that f_1 is continuous.

For $n \geqslant 1$ let Ω be the unit n-cube $[0,1]^n$. Let μ denote n-dimensional Lebesgue measure on Ω, let Σ consist of the μ-measurable subsets of Ω, and let $L_1 = L_1(\Omega, \Sigma, \mu)$. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are elements of Ω, we write $x \leqslant y$ if and only if $x_t \leqslant y_t$ for $1 \leqslant t \leqslant n$. A function g: $\Omega \to \mathbb{R}$ is said to be nondecreasing in each variable separately if $x, y \in \Omega$ and $x \leqslant y$ imply that $g(x) \leqslant g(y)$. Such a function is said to be *nondecreasing* if the following condition is also satisfied: If x is in the boundary of Ω, then $g(x) = \inf\{g(y): y \geqslant x\}$ if x has a zero coordinate, and $g(x) = \sup\{g(y): y \leqslant x\}$ otherwise. Let M consist of all nondecreasing functions. For f in L_1, let $\mu_1(f|M)$ denote the set of all best L_1-approximations to f by elements of M.

For $n = 1$ and f continuous, it was shown in **[1]** that there exists a unique best L_1-approximation f_1, to f by elements of M, and that f_1 is continuous. The purpose of the present note is to generalize this result to the case where n is any positive integer.

Our strategy includes the uniform approximation of f from above and below by step functions. For convenience we will use the dyadic rational partitions of Ω: For $k \geqslant 0$ let π_k denote the set of all points in Ω whose coordinates are rational numbers with denominator 2^k. The points of π_k divide Ω into a set of n-cubes $\{J(i)\}$, where each i has the form $i = (i_1, i_2, \ldots, i_n)$. We will henceforth call an n-cube a *cube*. The cubes $\{J(i)\}$ are pairwise disjoint, and their union is Ω: A cube in this collection contains the set

$$(x, y] = \{z: x_1 < z_1 \leqslant y_1, x_2 < z_2 \leqslant y_2, \ldots, x_n < z_n \leqslant y_n\}$$

and any part of the boundary of (x, y) which intersects the boundary of Ω.

We will also denote by π_k the set of all such cubes. Let the cubes in π_k be partially ordered by restricting the order $\leqslant$ on Ω to the set of all centers of cubes in π_k.

Let I_E denote the indicator function of a subset E of Ω, i.e., $I_E(x) = 1$ if $x \in E$ and $I_E(x) = 0$ otherwise, and let S_k consist of all functions f: $\Omega \to \mathbf{R}$ which have the form $f = \Sigma_i r_i I_{J(i)}$ where $\bigcup_i J(i) = \Omega$.

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If $f \in L_1$, then Lemma 3 in [2] states that $\mu_1(f|M) \neq \varnothing$ and, if $\bar{f} = \sup \mu_1(f|M)$ and $\underline{f} = \inf \mu_1(f|M)$, then $\bar{f}$ and $\underline{f}$ are in $\mu_1(f|M)$.

LEMMA 1. *If $f \in L_1$, then $\mu[\underline{f} < f < \bar{f}] = 0$.*

PROOF. The proof of (8) in [2] can be easily altered to show that $\mu[g_1 < f < g_2] = 0$ for any pair of elements g_1 and g_2 of $\mu_1(f|M)$.

LEMMA 2. *If $f \in S_k$, then $\underline{f}$ and $\bar{f}$ are (up to equivalence) also in S_k.*

PROOF. The argument for $\underline{f}$ is similar to that for $\bar{f}$, so we give only the latter. For each x in Ω, let $f^*(x)$ be the average value of $\bar{f}$ on the cube in π_k which contains x. Then $f^* \in M \cap S_k$. If J is a cube in π_k and $f > f^*$ on J, then

$$\int_J |f - f^*| = \int_J f - \int_J f^* = \int_J f - \mu J \cdot f^*(x)$$

$$= \int_J (f - \bar{f}) \leqslant \int_J |f - \bar{f}|.$$

Similarly, if $f < f^*$ on J, then $\int_J |f - f^*| \leqslant \int_J |f - \bar{f}|$. Thus $\|f - f^*\|_1 \leqslant \|f - \bar{f}\|_1$. Since $\bar{f} \in \mu_1(f|M)$, $\|f - f^*\|_1 = \|f - \bar{f}\|_1$, so $f^* \in \mu_1(f|M)$. If $\bar{f}$ is not constant on the interior of some cube in π_k, then there exists a set of positive measure on which $f^* > \bar{f}$, a contradiction. This concludes the proof of Lemma 2.

The previous lemma shows that, for f in S_k, any discussion involving $\underline{f}$ and $\bar{f}$ is essentially about best L_1-approximation of a function whose domain is a finite, partially ordered set. The following definitions will facilitate the investigation of this finite problem.

We will say that $J(i)$ and $J(j)$ in π_k are *adjacent* if $j = (i_1, i_2, \ldots, i_{t-1}, i_t \pm 1, i_{t+1}, \ldots, i_n)$ for some t, $1 \leqslant t \leqslant n$. The union of a set of cubes is said to be a *component* if, for any two cubes $J(i)$ and $J(j)$ in the set, there exist cubes $J(i^1) = J(i)$, $J(i^2), \ldots, J(i^m) = J(j)$ such that, for $1 \leqslant t \leqslant m - 1$, $J(i^t)$ is adjacent to $J(i^{t+1})$. If $g \in S_k$, let $g(j)$ be the value of g on $J(j)$. If $J(i)$ and $J(j)$ are any two adjacent cubes, we will call $|g(i) - g(j)|$ a *jump* of g.

LEMMA 3. *For any $\varepsilon > 0$, if $f \in S_k$ and f has no jump greater than ε, then neither $\underline{f}$ nor $\bar{f}$ has a jump greater then 3ε.*

PROOF. Suppose that $\bar{f}$ has a jump greater than 3ε (the proof for $\underline{f}$ is similar). We will derive a contradiction by constructing a pair of sets on one of which f has a jump greater than ε.

Suppose, without loss of generality, that

$$\bar{f}(i_1, i_2, \ldots, i_n) = 0$$

and

$$\bar{f}(i_1 + 1, i_2, \ldots, i_n) = 3\varepsilon.$$

Let

$$\begin{aligned}
A &= \{ j : j_1 = i_1, j_2 \geqslant i_2, \ldots, j_n \geqslant i_n \text{ and } \bar{f}(j) = 0 \},\\
B &= \{ j : j_1 = i_1 + 1, j_2 \leqslant i_2, \ldots, j_n \leqslant i_n \text{ and } \bar{f}(j) = 3\varepsilon \},\\
A' &= A \cup \{ (j_1 - 1, j_2, \ldots, j_n) : j \in B \},\\
B' &= B \cup \{ (j_1 + 1, j_2, \ldots, j_n) : j \in A \},\\
A^* &= \bigcup \{ J(j) : j \in A' \},\\
B^* &= \bigcup \{ J(j) : j \in B' \}.
\end{aligned}$$

Then $\mu A^* = \mu B^*$, and A^* and B^* each have exactly one component.

Since $\bar{f} \in \mu_1(f|M)$ and there exists $\delta > 0$ such that the function f^*, defined by

$$f^*(x) = \begin{cases} \bar{f}(x) + \delta, & x \in A^*,\\ \bar{f}(x), & x \notin A^*, \end{cases}$$

is nondecreasing, $\mu([f \leqslant \bar{f}] \cap A^*)/\mu A^* \geqslant 1/2$. Similarly, $\mu([f \geqslant \bar{f}] \cap B^*)/\mu B^* \geqslant 1/2$.

Since f has no jump greater than ε, for each cube $J(j)$ contained in $[f \geqslant \bar{f}] \cap B^*$, $f(j_1 - 1, j_2, \ldots, j_n) > \varepsilon$. Thus $f(j) \leqslant 0$ for half of the cubes $J(j)$ contained in A^*, and $f(j) > \varepsilon$ for the other half. Since A^* has only one component, there exist adjacent cubes $J(i)$ and $J(j)$ contained in A^* such that $f(i) = 0$ and $f(j) > \varepsilon$, a contradiction. This establishes Lemma 3.

LEMMA 4. *For any $\varepsilon > 0$, if $f \in S_k$ and f has no jump greater than ε, then $\bar{f} - \underline{f} \leqslant 4\varepsilon$.*

PROOF. Suppose, for contradiction, that A is a component of $[\bar{f} - \underline{f} > 4\varepsilon]$. Let $J(i)$ be a minimal element of the set of cubes contained in A. If $j \leqslant i$ and $J(j)$ is not contained in A but is adjacent to $J(i)$, then $\bar{f}(i) - \bar{f}(j) > 0$. Since there are a finite number of such $J(j)$'s, there exists $\delta > 0$ such that the function f^* in S_k, defined by

$$f^*(i) = \bar{f}(i) - \delta, \quad f^*(j) = \bar{f}(j), \qquad j \neq i,$$

is nondecreasing. Since there is no element of M L_1-closer to f than $\bar{f}$ is, we must conclude that $f(i) \geqslant \bar{f}(i)$. Clearly, the same conclusion holds if $J(i)$ is the smallest cube in π_k. Similarly, if $J(i)$ is a maximal element of A, then $f(i) \leqslant \underline{f}(i)$.

By Lemma 1 it is not possible that $\underline{f}(j) < f(j) < \bar{f}(j)$ for any j. Since A is a component, it contains adjacent cubes $J(i)$ and $J(j)$ such that $J(i) \leqslant J(j)$, $f(i) \geqslant \bar{f}(i)$, and $f(j) \leqslant \underline{f}(j)$. By Lemma 3, $\underline{f}(i) \geqslant \underline{f}(j) - 3\varepsilon$, so

$$f(i) \geqslant \bar{f}(i) > \underline{f}(i) + 4\varepsilon \geqslant \underline{f}(j) + \varepsilon \geqslant f(j) + \varepsilon,$$

which contradicts the hypothesis.

LEMMA 5. *Suppose $f, g \in L_1$. If $f \leqslant g$, then $\bar{f} \leqslant \bar{g}$ and $\underline{f} \leqslant \underline{g}$. For any constant c, $\overline{f + c} = \bar{f} + c$ and $\underline{f + c} = \underline{f} + c$.*

PROOF. The first statement is a particular case of Lemma 3 in [**2**], and the second statement is transparent.

THEOREM 6. *If $f\colon \Omega \to \mathbf{R}$ is continuous, then there exists a unique best L_1-approximation, f_1, to f by elements of M, and f_1 is continuous.*

PROOF. Let $\varepsilon > 0$ be given. Then there exists $k = k(\varepsilon) > 0$ such that $\sup_{x \in J} f(x) - \inf_{x \in J} f(x) < \varepsilon$ for every J in π_k. Define f^ε in S_k by

$$f^\varepsilon(x) = \sup_{x \in J} f(x), \qquad x \in J \in \pi_k,$$

and define f_ε similarly, with "sup" replaced by "inf". Since neither f_ε nor f^ε has a jump greater than ε, Lemma 4 implies that $\bar{f}_\varepsilon \leqslant \underline{f}_\varepsilon + 4\varepsilon$ and $\bar{f}^\varepsilon \leqslant \underline{f}^\varepsilon + 4\varepsilon$. By the choice of f_ε and f^ε, $f^\varepsilon - \varepsilon \leqslant f \leqslant f_\varepsilon + \varepsilon$, so Lemma 5 implies that

$$\bar{f} \leqslant \overline{f_\varepsilon + \varepsilon} = \bar{f}_\varepsilon + \varepsilon \leqslant \underline{f}_\varepsilon + 5\varepsilon \tag{1}$$

and

$$\underline{f} \geqslant \underline{f^\varepsilon - \varepsilon} = \underline{f}^\varepsilon - \varepsilon \geqslant \bar{f}^\varepsilon - 5\varepsilon. \tag{2}$$

Thus $\bar{f} - \underline{f} \leqslant \underline{f}_\varepsilon - \bar{f}^\varepsilon + 10\varepsilon < 10\varepsilon$. Since ε was arbitrary, we see that $\mu_1(f|M)$ is a singleton.

For continuity, note that $f \leqslant f^\varepsilon$, so $f_1 = \bar{f} \leqslant \bar{f}^\varepsilon$ and, similarly, $f_1 \geqslant \underline{f}_\varepsilon$. Since neither f_ε nor f^ε has a jump greater than ε, Lemma 3 implies that neither $\underline{f}_\varepsilon$ nor $\bar{f}^\varepsilon$ has a jump greater than 3ε. By (1) and (2), $\bar{f}^\varepsilon - \underline{f}_\varepsilon \leqslant 10\varepsilon$.

Let B be a ball in Ω of radius 2^{-k-1}. Then

$$\sup_{x \in B} f_1(x) - \inf_{x \in B} f_1(x) \leqslant 13n\varepsilon,$$

whence f_1 is continuous.

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