WEAK SPECTRAL THEORY

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Abstract. We initiate the weak spectrum of a linear operator on $L^p$ spaces, $1 < p < \infty$. The weak spectrum of a pseudo-differential operator with symbol in $S_{\rho,m}$, where $-\infty < \rho < \infty$ and $0 < \rho \leq 1$, is investigated.

1. Introduction. For $m \in (-\infty, \infty)$ and $\rho \in [0, 1]$, we define $S_{\rho,m}$ to be the set of all functions $\sigma$ in $C^\infty_0(R^n)$ such that, for each multi-index $\alpha$, $(D^\alpha \sigma)(\xi) = O(|\xi|^{m-\rho|\alpha|})$ as $|\xi| \to \infty$. Let $\sigma \in S_{\rho,m}$. Then we define the pseudo-differential operator $T_\sigma$, initially on $\mathcal{S}$ (the Schwartz space), by

$$
(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{R^n} e^{ix \cdot \xi} \sigma(\xi) \hat{\varphi}(\xi) \, d\xi
$$

for all $\varphi \in \mathcal{S}$. Here, $\hat{}$ denotes the Fourier transformation. Obviously $T_\sigma$ maps $\mathcal{S}$ into $\mathcal{S}$. It can be shown (see Proposition 2.1 in Wong [5]) that, for $1 \leq p \leq \infty$, $T_\sigma : \mathcal{S} \to \mathcal{S}$ is closable in $L^p(R^n)$. We denote the closure by $T_{op}$. Detailed information about the spectrum $\Sigma(T_{op})$ of $T_{op}$ can be found in Wong [3, 4, 5]. The corresponding results for partial differential operators have been gathered in Schechter [2].

2. The weak spectrum. Let $A$ be a closed linear operator defined on $L^p(R^n)$. Then a complex number $\lambda$ is said to be in the weak resolvent set $\rho(A)$ of $A$ if the range $R(A - \lambda)$ of $A - \lambda$ is dense in $L^p(R^n)$ and there is a constant $C > 0$ such that

$$
m \{ x \in R^n : |\varphi(x)| > \alpha \} \leq \left\{ C \| (A - \lambda) \varphi \| / \alpha \right\}^p
$$

for all $\alpha > 0$ and $\varphi$ in the domain $\mathcal{D}(A)$ of $A$. Here, $m \{ \cdots \}$ denotes the Lebesgue measure of $\{ \cdots \}$ and $\| \| \text{ the } L^p$ norm. As usual, the weak spectrum $\Sigma_w(A)$ of $A$ is defined to be $C - \rho_w(A)$. Obviously, $\Sigma_w(A) \subseteq \Sigma(A)$. That $\Sigma_w(A)$ can be a proper subset of $\Sigma(A)$ will be shown in §5.

3. On $\Sigma_w(T_{op})$, $1 < p < \infty$. We first show that $\Sigma_w(T_{op})$ is not empty.

Theorem 3.1. If $\sigma(\xi)$ is not bounded away from a complex number $\lambda$ for all $\xi \in R^n$, then $\lambda \in \Sigma_w(T_{op})$.

Proof. For simplicity, we suppose that $\lambda = 0$. Let $\{ \xi_k \}$ be a sequence of elements of $R^n$ such that $\sigma(\xi_k) \to 0$ as $k \to \infty$. Let $\{ \epsilon_k \}$ be a sequence of positive numbers. Let $\theta \in C^\infty_0(R^n)$ be such that $\theta(\xi_k) = 0$ for $|\xi_k| > 1$ and $(2\pi)^{-n/2} \int_{R^n} \theta(\xi) \, d\xi = 1$. Let
\( \psi \in \mathcal{S} \) be such that \( \hat{\psi} = \theta \). For \( k = 1, 2, \ldots \), define
\[
\varphi_k(x) = \epsilon_k^{n/p} \psi (\epsilon_k x) e^{i\epsilon_k x}.
\]
If \( 0 \in \rho_w(T_{\alpha p}) \), then there is a constant \( C > 0 \) such that
\[
m \left\{ x \in \mathbb{R}^n : |\varphi_k(x)| > \alpha \right\} \leq \left\{ C \|T \varphi_k\| / \alpha \right\}^p
\]
for all \( \alpha > 0 \) and \( k = 1, 2, \ldots \). Choosing \( \alpha = \frac{1}{2} \epsilon_k^{n/p} \) and using (3.1), inequality (3.2) becomes
\[
m \left\{ x \in \mathbb{R}^n : |\psi(\epsilon_k x)| > \frac{1}{2} \right\} \leq \epsilon_k^{-n} O\left( \|T \varphi_k\|^p \right).
\]
Since \( \psi (0) = (2\pi)^{-n/2} \int_{\mathbb{R}} \hat{\theta}(\xi) \, d\xi = 1 \), it follows that there exists a \( \delta > 0 \) such that \( |\psi(x)| > \frac{1}{2} \) whenever \( |x| \leq \delta \). Therefore
\[
m \left\{ x \in \mathbb{R}^n : |\psi(\epsilon_k x)| > \frac{1}{2} \right\} \geq \pi \left( \delta / \epsilon_k \right)^n
\]
for \( k = 1, 2, \ldots \). Hence, by (3.3),
\[
\pi \delta^n \leq O\left( \|T \varphi_k\|^p \right)
\]
for \( k = 1, 2, \ldots \). But as has been shown in the proof of Theorem 3.1 in Wong [5], we can choose the \( \epsilon_k \)'s going to zero so fast that \( \|T \varphi_k\| \to 0 \) as \( k \to \infty \). Thus (3.4) is impossible.

A useful consequence of Theorem 3.1 is

**Corollary 3.2.** \( \Sigma_w(T_{\alpha p}) \) contains the set \{ \( \sigma(\xi) : \xi \in \mathbb{R}^n \) \}.

4. Multipliers of weak type \((p, p)\), \(1 < p < \infty\). Let \( m \) be a bounded measurable function on \( \mathbb{R}^n \). For any \( \varphi \in \mathcal{S} \), we define \( T_m \varphi \) by
\[
(T_m \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\xi \cdot x} m(\xi) \varphi(\xi) \, d\xi.
\]
Suppose that there is a constant \( C > 0 \) such that
\[
m \left\{ x \in \mathbb{R}^n : |(T_m \varphi)(x)| > \alpha \right\} \leq \left\{ C \| \varphi \| / \alpha \right\}^p
\]
for all \( \alpha > 0 \) and \( \varphi \in \mathcal{S} \). Then we call \( T_m \) a multiplier of weak type \((p, p)\).

The connection between weak type multipliers and weak spectra is provided by

**Theorem 4.1.** A complex number \( \lambda \) is in the weak resolvent set \( \rho_w(T_{\alpha p}) \) of \( T_{\alpha p} \) if and only if \( 1 / (\alpha(\xi) - \lambda) \) is a multiplier of weak type \((p, p)\).

**Proof.** We first prove necessity. Again for simplicity, let \( \lambda = 0 \). By Theorem 3.1, \( \alpha(\xi) \) is bounded away from 0 for all \( \xi \in \mathbb{R}^n \). For any \( f \in \mathcal{S} \), define \( u \) by \( \hat{u}(\xi) = \hat{f}(\xi) / \alpha(\xi) \). Then \( u \in \mathcal{S} \) and \( T_\alpha u = f \). Since \( 0 \in \rho_w(T_{\alpha p}) \), it follows that there is a constant \( C > 0 \) such that
\[
m \left\{ x \in \mathbb{R}^n : |u(x)| > \alpha \right\} \leq \left\{ C \| f \| / \alpha \right\}^p
\]
for all \( \alpha > 0 \) and \( f \in \mathcal{S} \). Hence \( 1 / \alpha(\xi) \) is a multiplier of weak type \((p, p)\). Conversely, if \( 1 / \alpha(\xi) \) is a multiplier of weak type \((p, p)\), then there is a constant \( C > 0 \) such that
\[
m \left\{ x \in \mathbb{R}^n : |\varphi(\xi)| > \alpha \right\} \leq \left\{ C \| T_\alpha \varphi \| / \alpha \right\}^p
\]
for all \( \alpha > 0 \) and \( \varphi \in \mathcal{S} \). Since \( \sigma(\xi) \) is bounded away from 0 for all \( \xi \in \mathbb{R}^n \), it follows that \( \mathcal{S} \subseteq R(T_{\alpha p}) \). This proves that \( R(T_{\alpha p}) \) is dense in \( L^p(\mathbb{R}^n) \). Consequently, \( 0 \in \rho_{\alpha}(T_{\alpha p}) \) if we can show that (4.1) is valid for all \( \varphi \in \mathcal{D}(T_{\alpha p}) \).

**Lemma 4.2.** Inequality (4.1) is valid for all \( \varphi \in \mathcal{D}(T_{\alpha p}) \).

**Proof.** For any \( \varphi \in \mathcal{D}(T_{\alpha p}) \), let \( \{ \varphi_k \} \) be a sequence of functions in \( C_0^\infty(\mathbb{R}^n) \) such that \( \varphi_k \to \varphi \) and \( T\varphi_k \to T\varphi \) in \( L^p(\mathbb{R}^n) \) as \( k \to \infty \). Pick a subsequence of \( \{ \varphi_k \} \), again denoted by \( \{ \varphi_k \} \), such that \( \varphi_k \to \varphi \) a.e. on \( \mathbb{R}^n \). For any \( \alpha > 0 \), we set

\[
E_\alpha = \{ x \in \mathbb{R}^n : |\varphi_k(x)| > \alpha \}
\]

and

\[
E_k(\alpha) = \{ x \in \mathbb{R}^n : |\varphi_k(x)| > \alpha \}
\]

for \( k = 1, 2, \ldots \). Since \( \varphi \in L^p(\mathbb{R}^n) \), it follows that \( m(E_\alpha) < \infty \). So for any \( \varepsilon > 0 \), by Egoroff’s Theorem, we can find a measurable set \( A_\varepsilon \subseteq \mathbb{R}^n \) such that \( m(A_\varepsilon) < \varepsilon \) and \( \varphi_k(x) \to \varphi(x) \) uniformly for all \( x \in E(\alpha) - A_\varepsilon \). Hence, there exists a positive integer \( K \) such that \( |\varphi_k(x)| > \alpha \) whenever \( k \geq K \). For such \( k \)'s, \( E(\alpha) - A_\varepsilon \subseteq E_k(\alpha) \), and using (4.1) and letting \( k \to \infty \), we get

\[
m(E_\alpha) - \varepsilon \leq \left( C\|T_{\alpha p}\|/\alpha \right)^p
\]

for every \( \alpha > 0 \). Since \( \varepsilon \) is an arbitrary positive number, the proof is complete.

5. **An example.** We begin with an observation.

**Lemma 5.1.** For any \( p \) such that \( 1 \leq p < \infty \), a sufficient condition for \( \Sigma_w(T_{\alpha p}) = \Sigma(T_{\alpha p}) \) is that \( \Sigma(T_{\alpha p}) = \{ \sigma(\xi) : \xi \in \mathbb{R}^n \} \).

Lemma 5.1 follows immediately from Corollary 3.2 and the fact that \( \Sigma_w(T_{\alpha p}) \subseteq \Sigma(T_{\alpha p}) \).

Let \( \tau \) be the function defined by \( \tau(\xi) = e^{i|\xi|^a}/(1 + |\xi|^c) \), where \( 0 < a < 1 \) and \( 0 < c < na/2 \). Then, defining \( \sigma \) by \( \sigma(\xi) = 1/\tau(\xi) \), it is clear that \( \sigma \in S_1(\xi, 0) \). As has been proved in Wong [3, 4], \( \Sigma(T_{\alpha p}) = \{ \sigma(\xi) : \xi \in \mathbb{R}^n \} \) if \( p \) is any number such that \( 1 < p < \infty \) and \( |1/p - 1/2| < c/na \), and \( \Sigma(T_{\alpha p}) = \{ C \} \) if \( |1/p - 1/2| > c/na \). The following result tells us that we know exactly what \( \Sigma_w(T_{\alpha p}) \) is if \( 1 < p < \infty \).

**Theorem 5.2.** \( \Sigma_w(T_{\alpha p}) = \Sigma(T_{\alpha p}), \) \( 1 < p < \infty \).

**Proof.** In view of Lemma 5.1, we need only consider \( |1/p - 1/2| > c/na \). We first suppose that \( 1/p > 1/2 + c/na \). If \( \lambda \in \Sigma(T_{\alpha p}) \) and \( \lambda \not\in \Sigma_w(T_{\alpha p}) \), then by Theorem 4.1, \( 1/(\sigma(\xi) - \lambda) \) is a multiplier of weak types \( (2, 2) \) and \( (p, p) \). Hence, by the Marcinkiewicz interpolation theorem, \( 1/(\sigma(\xi) - \lambda) \) is an \( L^q \) multiplier for any \( q \) such that \( 1/2 + c/na < 1/q < 1/p \). Thus, by Theorem 3.3 in Wong [3], \( \lambda \in \rho(T_{\alpha q}) \).

But Theorem 3.1 in Wong [4] says that the spectrum of \( T_{\alpha q} \) is either \( \mathbb{C} \) or \( \{ \sigma(\xi) : \xi \in \mathbb{R}^n \} \). Hence, \( \Sigma(T_{\alpha q}) = \{ \sigma(\xi) : \xi \in \mathbb{R}^n \} \). This is a contradiction. The proof for the case when \( 1/p < 1/2 - c/na \) is similar.

**Theorem 5.3.** \( \Sigma_w(T_{\alpha q}) = \{ \sigma(\xi) : \xi \in \mathbb{R}^n \} \).
PROOF. By Theorems 3.1 and 4.1, we need only show that if $\lambda \in \mathbb{C}$ is such that $\sigma(\xi) \neq \lambda$ for all $\xi \in \mathbb{R}^n$, $1/(\sigma(\xi) - \lambda)$ is a multiplier of weak type $(1, 1)$. But an easy computation shows that $1/(\sigma(\xi) - \lambda) \in S_{1-\alpha,0}^1$, and hence it follows from Theorem 2' in Fefferman [1] that $1/(\sigma(\xi) - \lambda)$ is indeed a multiplier of weak type $(1, 1)$.

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