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A NONLINEAR BOUNDARY PROBLEM

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ABSTRACT. A nonlinear Hilbert problem of power type is solved in closed form by representing a sectionally holomorphic function by means of an integral with power kernel. This technique transforms the problem to one of solving an integral equation of the generalized Abel type.

1. Introduction. In this paper we consider the problem of finding a function $\Phi(z) = u + iv$, holomorphic in the plane cut along the interval $[0,1]$ and vanishing at infinity, such that

$$[\Phi^+(x)]^\alpha + [\Phi^-(x)]^\alpha = f \quad \text{on } (0,1), \qquad 0 < \alpha < 1, \tag{1.1}$$

where, as usual, $\Phi^{\pm}(x)$ are the limiting values of $\Phi(z)$ on the approaches to the cut from above and below, respectively. As basis for the presented process of solution it is assumed that $f(x) \in D(H)$. The class $D(H)$ is the union of all functions $f(x)$ with a derivative satisfying a Hölder condition on $[0,1]$ with the possible exception of the endpoints 0, 1, but, for $x \in (0,1)$ and near p,

$$|f'(x)| \leqslant M/|x-p|^\mu;$$

p stands for either of the endpoints, and M, μ are positive constants, $\mu < 1$.

It appears that the nonlinear boundary problem (1.1) is related in some manner to an ordinary linear nonhomogeneous Hilbert problem. However, we avoid this approach to the solution because Φ^α may not be analytic. Our procedure then is to put

$$\Phi(z) = \left(\int_0^1 (t-z)^{-\alpha}\varphi(t)\,dt\right)^{1/\alpha}, \qquad z \notin [0,1], \tag{1.2}$$

where the function φ is sought in the class defined by

$$\varphi(x) = \frac{\varphi^*(x)}{x^{1-\varepsilon_1}(1-x)^{1-\varepsilon_2}}, \qquad 0 < x < 1, \tag{1.3}$$

with $\varepsilon_1, \varepsilon_2 > 0$ and $\varphi^*(x)$ is Hölder continuous on $[0,1]$.

The limiting values $\Phi^{\pm}(x)$ of $\Phi(z)$ on the approaches to the cut from above and below will be needed. Consequently, we define

$$\arg(t-z) \to \mp\pi, \quad z \to x \pm i0, \qquad 0 \leqslant t < x \leqslant 1,$$
$$\arg(t-z) \to 0, \quad z \to x \pm i0, \qquad 0 \leqslant x < t \leqslant 1.$$

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Then it follows from (1.2) that

$$\Phi^{\pm}(x) = \left[J_{1\alpha}(x)e^{\pm i\pi\alpha} + J_{2\alpha}(x)\right]^{1/\alpha}, \tag{1.4}$$

where

$$J_{1\alpha}(x) = \int_0^x (x-t)^{-\alpha}\varphi(t)\,dt, \qquad J_{2\alpha}(x) = \int_x^1 (t-x)^{-\alpha}\varphi(t)\,dt.$$

Thus, by substituting into (1.1) the limiting values $\Phi^{\pm}(x)$ from formulae (1.4), we obtain

$$2J_{1\alpha}(x)\cos\pi\alpha + 2J_{2\alpha}(x) = f(x). \tag{1.5}$$

Equation (1.5) is a special case of the type known as Abel's generalized integral equation [**1**]. Its solution in the class defined by (1.3) determines the function $\Phi(z)$ given by (1.2).

2. Solution of the generalized Abel equation. Consider the sectionally holomorphic function Ω defined by

$$\Omega(z) = [z(1-z)]^{(\alpha-1)/2}\int_0^1 (t-z)^{-\alpha}\varphi(t)\,dt, \qquad z \notin [0,1],$$

where some branch of the many-valued function $[z(1-z)]^{(\alpha-1)/2}(t-z)^{-\alpha}$ is chosen.

Define $\arg z \to 0$ as $z \to x + i0$ and $\arg z \to 2\pi$ as $z \to x - i0$ for $0 < x < 1$. Then it follows that

$$\begin{cases} R(x)\Omega^{+}(x) = J_{1\alpha}(x)e^{i\pi\alpha} + J_{2\alpha}(x), \\ -R(x)\Omega^{-}(x) = J_{1\alpha}(x) + J_{2\alpha}(x)e^{i\pi\alpha}, \end{cases} \tag{2.1}$$

where $R(x) = [x(1-x)]$.

Solving (2.1) for $J_{1\alpha}(x)$, $J_{2\alpha}(x)$ and inserting these values into (1.5), we obtain the nonhomogeneous Hilbert problem $\Omega^{+}(x) = e^{-i\pi\alpha}\Omega^{-}(x) + f(x)/R(x)$ or, equivalently,

$$\frac{\Omega^{+}(x)}{H^{+}(x)} - \frac{\Omega^{-}(x)}{H^{-}(x)} = \frac{f(x)}{R(x)H^{+}(x)},$$

where $H(z) = z^{\alpha/2}(1-z)^{-\alpha/2}$. Thus, we have

$$\Omega(z) = \frac{H(z)}{2\pi i}\int_0^1 \frac{f(t)\,dt}{H^{+}(t)R(t)(t-z)},$$

so that by the well-known Plemelj formulae [**2**] for the limiting values of a Cauchy integral, we have

$$\Omega^{\pm}(x) = H^{\pm}(x)\left(\frac{\pm f(x)}{2H^{+}(x)R(x)} + \frac{1}{2\pi i}\int_0^1 \frac{f(x)\,dt}{H^{+}(t)R(t)(t-x)}\right). \tag{2.2}$$

Now, by virtue of (2.1) and (2.2), it follows that

$$
\begin{aligned}
(2.3) \qquad J_{1\alpha}(x) &= \frac{R(x)\left[\Omega^+(x)e^{i\pi\alpha} + \Omega^-(x)\right]}{e^{2\pi i\alpha} - 1} \\
&= \frac{-R(x)H^+(x)}{2\pi \sin \pi\alpha} \int_0^1 \frac{f(t)\,dt}{H^+(t)R(t)(t-x)} \\
&= \frac{-\sqrt{x}(1-x)^{1/2-\alpha}}{2\pi \sin \pi\alpha} \int_0^1 \frac{(1-t)^\alpha f(t)\,dt}{\sqrt{t}\sqrt{1-t}(t-x)}.
\end{aligned}
$$

Equation (2.3) is an ordinary Abel equation to be solved for φ. Because the solution depends on the differentiability of the right-hand side of (2.3), we establish the following lemma.

LEMMA. *For $f(x) \in D(H)$ and $0 < \alpha < 1$, let*

$$Q(x) = \sqrt{x}\sqrt{1-x} \int_0^1 \frac{(1-t)^\alpha f(t)\,dt}{\sqrt{t}\sqrt{1-t}(t-x)}, \qquad 0 < x < 1,$$

where the integral is taken in the sense of principal value. Then $Q(x)$ is differentiable on $0 < x < 1$ and

$$Q'(x) = \frac{1}{\sqrt{x}\sqrt{1-x}} \int_0^1 \frac{\sqrt{t}\sqrt{1-t}\left[(1-t)^\alpha f(t)\right]'\,dt}{t-x}.$$

PROOF. For sufficiently small $\varepsilon > 0$ put

$$I_\varepsilon(x) = \int_0^{x-\varepsilon} \frac{(1-t)^\alpha f(t)\,dt}{\sqrt{t}\sqrt{1-t}(t-x)} + \int_{x+\varepsilon}^1 \frac{(1-t)^\alpha f(t)\,dt}{\sqrt{t}\sqrt{1-t}(t-x)}.$$

Also let

$$A(x,t) = \log\left|\frac{\sqrt{x}\sqrt{1-t} - \sqrt{t}\sqrt{1-x}}{\sqrt{x}\sqrt{1-t} + \sqrt{t}\sqrt{1-x}}\right|, \qquad t \neq x.$$

Then for fixed x and variable t we have

$$\frac{dA(x,t)}{\sqrt{x}\sqrt{1-x}} = \frac{dt}{\sqrt{t}\sqrt{1-t}(t-x)},$$

and it follows from integration by parts that

$$
\begin{aligned}
&\sqrt{x}\sqrt{1-x}\,I_\varepsilon(x) \\
&\quad = (1-x+\varepsilon)^\alpha f(x-\varepsilon)A(x,x-\varepsilon) - (1-x-\varepsilon)^\alpha f(x+\varepsilon)A(x,x+\varepsilon) \\
&\qquad - \int_0^{x-\varepsilon} A(x,t)\left[(1-t)^\alpha f(t)\right]'\,dt - \int_{x+\varepsilon}^1 A(x,t)\left[(1-t)^\alpha f(t)\right]'\,dt.
\end{aligned}
$$

Recall the meaning of $A(x,t)$, and then rationalize and rearrange some terms to find that

$$
\begin{aligned}
&(1-x+\varepsilon)^\alpha f(x-\varepsilon)A(x,x-\varepsilon) - (1-x-\varepsilon)^\alpha f(x+\varepsilon)A(x,x+\varepsilon) \\
&\qquad = 2(1-x-\varepsilon)^\alpha f(x+\varepsilon)\log(\sqrt{x}\sqrt{1-x-\varepsilon} + \sqrt{x+\varepsilon}\sqrt{1-x}) \\
&\qquad\quad - 2(1-x+\varepsilon)^\alpha f(x-\varepsilon)\log(\sqrt{x}\sqrt{1-x+\varepsilon} + \sqrt{x-\varepsilon}\sqrt{1-x}) \\
&\qquad\quad + \left[(1-x+\varepsilon)^\alpha f(x-\varepsilon) - (1-x-\varepsilon)^\alpha f(x+\varepsilon)\right]\log\varepsilon.
\end{aligned}
$$

Thus it follows that

$$\begin{aligned}
Q(x) &= \sqrt{x}\sqrt{1-x}\,\lim_{\varepsilon\to 0} I_\varepsilon(x)\\
&= -\int_0^1 A(x,t)\left[(1-t)^\alpha f(t)\right]' dt\\
&= \int_0^1 \left[(1-t)^\alpha f(t)\right]' \log\left|\frac{\sqrt{x}\sqrt{1-t}+\sqrt{t}\sqrt{1-x}}{\sqrt{x}\sqrt{1-t}-\sqrt{t}\sqrt{1-x}}\right| dt
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
Q(x) &= 2\int_0^1 \left[(1-t)^\alpha f(t)\right]' \log(\sqrt{x}\sqrt{1-t}+\sqrt{t}\sqrt{1-x})\, dt \\
&\quad -\int_0^1 \left[(1-t)^\alpha f(t)\right]' \log|t-x|\, dt.
\end{aligned} \tag{2.4}$$

The second integral in (2.4) is understood in the sense of principal value. Denote this integral by $J(x)$ and put

$$\begin{aligned}
J_\varepsilon(x) &= \int_0^{x-\varepsilon} \left[(1-t)^\alpha f(t)\right]' \log|t-x|\, dt\\
&\quad + \int_{x+\varepsilon}^1 \left[(1-t)^\alpha f(t)\right]' \log|t-x|\, dt,
\end{aligned}$$

where $\varepsilon > 0$ is sufficiently small. Now

$$\begin{aligned}
J_\varepsilon'(x) = \Big[&(1-x+\varepsilon)^\alpha f'(x-\varepsilon) - (1-x-\varepsilon)^\alpha f'(x+\varepsilon)\\
&+\alpha(1-x-\varepsilon)^{\alpha-1} f(x+\varepsilon) - \alpha(1-x+\varepsilon)^{\alpha-1} f(x-\varepsilon)\Big]\log\varepsilon\\
&-\int_0^{x-\varepsilon} \frac{\left[(1-t)^\alpha f(t)\right]'}{t-x}\, dt - \int_{x+\varepsilon}^1 \frac{\left[(1-t)^\alpha f(t)\right]'}{t-x}\, dt,
\end{aligned}$$

and hence $J_\varepsilon'(x)$ converges uniformly to the limit

$$\lim_{\varepsilon\to 0} J_\varepsilon'(x) = -\int_0^1 \frac{\left[(1-t)^\alpha f(t)\right]'}{t-x}\, dt.$$

Let this limit be denoted by $L(x)$. Now for any $x \in (0,1)$, pick x_0 such that $0 < x_0 < x$. Thus, since $J_\varepsilon'(x)$ converges uniformly to $L(x)$, we have

$$\int_{x_0}^x L(t)\, dt = \lim_{\varepsilon\to 0}\int_{x_0}^x J_\varepsilon'(t)\, dt = J(x) - J(x_0).$$

But $L(x)$ is continuous, and therefore $J'(x)$ exists and

$$J'(x) = L(x) = -\int_0^1 \frac{\left[(1-t)^\alpha f(t)\right]'}{t-x}\, dt.$$

Consequently, we have

$$Q'(x) = \int_0^1 \frac{\left(\sqrt{1-t}/\sqrt{x} - \sqrt{t}/\sqrt{1-x}\right)\left[(1-t)^\alpha f(t)\right]'}{\sqrt{x}\sqrt{1-t} + \sqrt{t}\sqrt{1-x}}\,dt$$

$$+ \int_0^1 \frac{\left[(1-t)^\alpha f(t)\right]'}{t-x}\,dt$$

$$= \int_0^1 \frac{\left(\sqrt{1-t}/\sqrt{x} - \sqrt{t}/\sqrt{1-x}\right)\left(\sqrt{x}\sqrt{1-t} - \sqrt{t}\sqrt{1-x}\right)\left[(1-t)^\alpha f(t)\right]'}{x-t}\,dt$$

$$+ \int_0^1 \frac{\left[(1-t)^\alpha f(t)\right]'}{t-x}\,dt$$

$$= \frac{1}{\sqrt{x}\sqrt{1-x}} \int_0^1 \frac{\sqrt{t}\sqrt{1-t}\left[(1-t)^\alpha f(t)\right]'}{t-x}\,dt.$$

This completes the proof.

Let us now find the solution φ of the ordinary Abel equation (2.3). First note that (2.3) is equivalent to

$$\int_0^x \frac{\varphi(t)}{(x-t)^\alpha}\,dt = \frac{-Q(x)}{2\pi(1-x)^\alpha \sin \pi\alpha},$$

so that

$$\varphi(x) = -\frac{1}{2\pi^2}\frac{d}{dx}\int_0^x \frac{Q(t)\,dt}{(1-t)^\alpha (x-t)^{1-\alpha}}.$$

Now integrate by parts and note that $Q(0) = 0$ by (2.4). Consequently, we have

$$\varphi(x) = \frac{-1}{2\pi^2\alpha}\frac{d}{dx}\int_0^x (x-t)^\alpha \left[(1-t)^{-\alpha} Q(t)\right]'\,dt$$

$$= -\frac{1}{2\pi^2}\int_0^x \frac{\left[(1-t)^\alpha Q(t)\right]'}{(x-t)^{1-\alpha}}\,dt$$

or, equivalently,

$$\varphi(x) = -\frac{1}{2\pi^2}\int_0^x \frac{P(t)\,dt}{(x-t)^{1-\alpha}(1-t)^{1+\alpha}}, \tag{2.5}$$

where $P(t) = \alpha Q(t) + (1-t)Q'(t)$.

In order to express $P(t)$ in terms of the function f, we apply the lemma pertaining to the differentiability of $Q(x)$. Thus we find that

$$P(x) = \alpha\sqrt{x}\sqrt{1-x}\int_0^1 \frac{(1-t)^\alpha f(t)\,dt}{\sqrt{t}\sqrt{1-t}\,(t-x)}$$

$$+ \frac{\sqrt{1-x}}{\sqrt{x}}\int_0^1 \frac{\sqrt{t}\sqrt{1-t}\left[(1-t)^\alpha f(t)\right]'\,dt}{t-x}$$

and, after performing the indicated differentiation of the quantity in the integrand of the second integral, it follows that

$$(2.6)\qquad P(x)=\frac{\sqrt{1-x}}{\sqrt{x}}\left(\int_0^1\frac{\sqrt{t}\,(1-t)^{\alpha+1/2}f'(t)\,dt}{t-x}-\alpha C\right),$$

where

$$(2.7)\qquad C=\int_0^1 t^{-1/2}(1-t)^{\alpha-1/2}f(t)\,dt.$$

Hence, by virtue of (2.5) and (2.6), we have a formula for the function φ. Therefore our solution of (1.1) is given by (1.2) with φ as in (2.5).

We may now formulate the following theorem.

THEOREM. *For* $0<\alpha<1$ *and* $f(x)\in D(H)$, *let*

$$F(x)=\int_0^1\frac{\sqrt{t}\,(1-t)^{\alpha+1/2}f'(t)\,dt}{x-t}\quad\text{and}\quad C=\int_0^1 t^{-1/2}(1-t)^{\alpha-1/2}f(t)\,dt.$$

Then the sectionally holomorphic function

$$\Phi(z)=\left(\int_0^1(t-z)^{-\alpha}\varphi(t)\,dt\right)^{1/\alpha},\qquad z\notin[0,1],$$

where

$$\varphi(x)=\frac{1}{2\pi^2}\int_0^x\frac{F(t)\,dt}{\sqrt{t}\,(1-t)^{\alpha+1/2}(x-t)^{1-\alpha}}+\frac{\alpha C\Gamma(\alpha)x^{\alpha-1/2}}{2\pi^{3/2}\Gamma(\alpha+1/2)\sqrt{1-x}},$$

solves the nonlinear boundary problem $(\Phi^+(x))^\alpha+(\Phi^-(x))^\alpha=f(x)$, $0<x<1$.

EXAMPLE. Given that $\alpha=\frac{3}{4}$ and $f(x)=32x^{1/4}(1-x)+8\sqrt{2}\,(1-x)^{1/4}(4x-3)+1$, find $\Phi(z)$ satisfying (1.1).

Solution. The reader will not encounter any difficulty in verifying that

$$\int_0^1\frac{\sqrt{t}\sqrt{1-t}}{t-x}dt=\frac{\pi}{2}(1-2x),$$

$$\int_0^1\left(\frac{1-t}{t}\right)^{1/4}\frac{dt}{t-x}=\pi\left(\frac{1-x}{x}\right)^{1/4}-\pi\sqrt{2}$$

for $0<x<1$. An excellent method for evaluating these integrals is found in Levinson's paper [**3**]. These results will be used to evaluate the singular integral

$$F(x)=\int_0^1\frac{\sqrt{t}\,(1-t)^{5/4}f'(t)}{x-t}dt.$$

We find that

$$F(x)=\int_0^1\left[8\left(\frac{1-t}{t}\right)^{1/4}(1-6t+5t^2)+2\sqrt{2}\sqrt{t}\sqrt{1-t}(19-20t)\right]\frac{dt}{x-t}$$
$$=\int_0^1\left[8\left(\frac{1-t}{t}\right)^{1/4}\left(6-5x-5t-\frac{5x^2-6x+1}{t-x}\right)\right.$$
$$\left.+2\sqrt{2}\sqrt{t}\sqrt{1-t}\left(20-\frac{19-20x}{t-x}\right)\right]dt$$
$$=\frac{9\pi\sqrt{2}}{4}-8\pi(1-6x+5x^2)\left(\frac{1-x}{x}\right)^{1/4}.$$

Also the constant C defined by (2.7) is

$$C=\int_0^1 t^{-1/2}(1-t)^{1/4}f(t)\,dt=-3\pi\sqrt{2}+\frac{\Gamma^2(1/4)}{3\sqrt{2\pi}},$$

and hence the formula for $\varphi(x)$ in the theorem implies $\varphi(x)=\sqrt{2}(5x-4)+x^{1/4}/(2\pi\sqrt{1-x})$. Thus it follows from (1.2) that

$$\Phi(z)=\left[4\sqrt{2}(1-z)^{1/4}(4z-3)+16\sqrt{2}(-z)^{1/4}(1-z)\right.$$
$$\left.+\int_0^1\frac{t^{1/4}\,dt}{2\pi\sqrt{1-t}(t-z)^{3/4}}\right]^{4/3}.$$

The fact that this function solves (1.1) can be verified by direct substitution. For instance,

$$(\Phi^+(x))^{3/4}+(\Phi^-(x))^{3/4}=32x^{1/4}(1-x)+8\sqrt{2}(1-x)^{1/4}(4x-3)+K,$$

where

$$K=\frac{-\sqrt{2}}{2\pi}\int_0^x\frac{t^{1/4}\,dt}{\sqrt{1-t}(x-t)^{3/4}}+\frac{1}{\pi}\int_x^1\frac{t^{1/4}\,dt}{\sqrt{1-t}(t-x)^{3/4}}.$$

We now show that $K=1$ by using the following formulae for the hypergeometric function **[4]**:

(2.8)

$$\begin{cases}F(a,b;c;x)=(1-x)^{c-a-b}F(c-a,c-b;c;x),\\ F(a,b;c;x)=\dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-x)\\ \qquad+(1-x)^{c-a-b}\dfrac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b;c-a-b+1;1-x).\end{cases}$$

Now it is easy to see by virtue of a variable change from t to u that

$$K = -\frac{\sqrt{2x}}{2\pi}\int_0^1 u^{1/4}(1-u)^{-3/4}(1-xu)^{-1/2}\,du + \frac{(1-x)^{-1/4}}{\pi}\int_0^1 \frac{[1-(1-x)u]^{1/4}}{\sqrt{u}(1-u)^{3/4}}\,du$$

$$= \frac{\sqrt{2}\,\Gamma^2(1/4)}{4\pi^{3/2}}\left[-\sqrt{x}\,F\left(\frac{1}{2},\frac{5}{4};\frac{3}{2};x\right) + 2(1-x)^{-1/4}F\left(-\frac{1}{4},\frac{1}{2};\frac{3}{4};1-x\right)\right]$$

$$= \frac{\sqrt{2}\,\Gamma^2(1/4)}{4\pi^{3/2}}\left[-\frac{\sqrt{\pi}\,\Gamma(-1/4)}{2\Gamma(1/4)} - 2\sqrt{x}(1-x)^{-1/4}F\left(1,\frac{1}{4};\frac{3}{4};1-x\right) + 2(1-x)^{-1/4}F\left(-\frac{1}{4};\frac{1}{2};\frac{3}{2};1-x\right)\right]$$

$$= 1 + \frac{\sqrt{2}\,\Gamma^2(1/4)}{2\pi^{3/2}}(1-x)^{-1/4}\left[F\left(-\frac{1}{4};\frac{1}{2};\frac{3}{4};1-x\right) - \sqrt{x}\,F\left(1,\frac{1}{4};\frac{3}{4};1-x\right)\right].$$

But the quantity within the brackets vanishes by virtue of the first formula in (2.8); hence $K = 1$ and the solution is verified.

3. A power type boundary problem with a variable coefficient. The boundary problem

$$(\Phi^+(x))^\alpha + G(x)(\Phi^-(x))^\alpha = f(x) \quad \text{on } (0,1) \text{ with } 0 < \alpha < 1 \tag{3.1}$$

can be put in the form (1.1) by using the sectionally holomorphic function X defined by

$$X(z) = \exp\left(\frac{1}{2\pi i\alpha}\int_0^1 \frac{\log G(t)}{t-z}\,dt\right), \qquad z \notin [0,1].$$

It is assumed that the Hölder continuous function $G(x) \neq 0$. The function $X(z)$ is a solution of the homogeneous boundary problem $(X^+(x))^\alpha = G(x)(X^-(x))^\alpha$, so that (3.1) becomes

$$\left(\frac{\Phi^+(x)}{X^+(x)}\right)^\alpha + \left(\frac{\Phi^-(x)}{X^-(x)}\right)^\alpha = \frac{f(x)}{(X^+(x))^\alpha},$$

and the theory for (1.1) applies provided the function $f(x)(X^+(x))^{-\alpha}$ belongs to the class $D(H)$.

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