

SPHERICAL QUADRATURE AND INVERSION OF RADON TRANSFORMS

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ABSTRACT. An equivalence is established between certain aspects of: (i) mechanical quadrature for integration on a sphere; (ii) a ridge function representation problem connected with inversion of Radon transform data.

This paper is concerned with the functional equation

$$(1) \quad G(x) = \sum_{i=1}^N h_i(\langle x, u_i \rangle), \quad |x| < 1.$$

Here x is a variable point in d -dimensional space R^d ($d \geq 2$), u_1, \dots, u_N are given unit vectors in R^d , $\langle x, u_i \rangle$ denotes the usual inner (dot) product, $|x| = \langle x, x \rangle^{1/2}$, and $h_1(t), \dots, h_N(t)$ are real-valued functions on the interval $-1 < t < 1$.

The reason for our interest in (1) is its connection with the convolution method for inverting Radon transform data, for example see [2, 5, 8, 9]. For that application, the h_i 's should be chosen so that G is a convolution summability kernel (in other words, an approximate Dirac delta function) in a neighborhood of the origin. Following [3] we call the terms on the right-hand side of (1) ridge functions.

Because of results in [4 and 6], we are particularly interested in cases where $G(x)$ depends only on $|x|$. In such cases, there is a close relation between the ridge function representation (1) and mechanical quadrature on the sphere in R^d ; see Theorem 1. To put off certain technicalities, we have confined Theorem 1 to the case where G and h_1, \dots, h_N are assumed to be polynomials. As shown by Theorem 3, this assumption is redundant. For background on mechanical quadrature for spherical integration see [7 and 10]; here we use the notation S_{d-1} for the unit sphere $|x| = 1$ in R^d and take $\omega_{d-1} = \int_{S_{d-1}} d\omega(u)$ where $d\omega(u)$ denotes the usual Lebesgue measure on S_{d-1} .

THEOREM 1. *A necessary and sufficient condition for a radial polynomial G to enjoy a representation (1) with polynomials for h_1, \dots, h_N is the existence of constants w_1, \dots, w_N (quadrature weights) that make*

$$(2) \quad \frac{1}{\omega_{d-1}} \int_{S_{d-1}} p(u) d\omega(u) = \sum_{i=1}^N w_i p(u_i)$$

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valid for all even polynomials p of degree $\leq \deg G$. In that case (1) is satisfied with $h_i = w_i h_G$, where $h_G(t) = \sum_{n=0}^{(\deg G)/2} (g_n/\mu_n) t^{2n}$. Here g_n denotes the coefficient of $|x|^{2n}$ in $G(x)$ and μ_n is the constant defined by

$$(3) \quad \frac{1}{\omega_{d-1}} \int_{S_{d-1}} \langle x, u \rangle^{2n} d\omega(u) = \mu_n |x|^{2n}.$$

REMARKS. In the case $d = 2$ the relationship between the directions, u_1, \dots, u_N and the maximum degree, call it $\nu = \nu(u_1, \dots, u_N)$, of the radial polynomials G which enjoy representation (1) was studied in [6]. The above theorem shows that this same number ν gives the maximum degree of even polynomials p for which the exact mechanical quadrature formula (2) holds on the circle.

The fact that (2) is sufficient to imply (1) for radial polynomials is implicit in the calculations of §5 in [2].

As mentioned above, it need not be assumed that G and h_1, \dots, h_N are polynomials; indeed, Theorem 3 and its corollary show that it suffices for (1) to hold in a distributional sense.

Applying (2) with $p = h_G$ gives

$$(4) \quad \int_{S_{d-1}} h_G(\langle x, u \rangle) d\omega(u) = \omega_{d-1} G(x)$$

which may be written in the equivalent form

$$(5) \quad \omega_{d-2} \int_{-r}^r h_G(t) \left(1 - \frac{t^2}{r^2}\right)^{(d-3)/2} \frac{1}{r} dt = \omega_{d-1} G(re).$$

where $r = |x|$ and $x = re$. Regarding (5) as an integral equation for h_G provides an alternative characterization of h_G .

Using (5) and standard notation for the Gegenbauer and the Jacobi polynomials, it may be verified (e.g. make the change of variable $t = \gamma\sqrt{x}$ in [1, §13.1, entry (41)] and take $\nu = 1/2$, $\mu = (d - 1)/2$, $\gamma = \sqrt{r}$) that $h_G(t) = C_{2n}^\lambda(t)$, $\lambda = \alpha + d/2$, $\alpha > -1$, corresponds to $G(x) = c_n P_n^{(\alpha, \beta)}(2|x|^2 - 1)$, $\beta = (d - 2)/2$. Here c_n is a constant that can be determined by taking $x = 0$ in (4). These orthogonal polynomials are very convenient bases for numerical work involving G and h_G . They are also useful (see [5]) for analyzing effects of noisy data.

PROOF. For sufficiency, take $p(u) = \langle x, u \rangle^{2n}$ in (2) and combine with (3) to obtain

$$\mu_n |x|^{2n} = \sum_{i=1}^N w_i \langle x, u_i \rangle^{2n}$$

for $n = 0, \dots, (\deg G)/2$. From this it is evident that $h_i = w_i h_G$ satisfies (1).

To prove necessity, let $2k$ be the degree of G . (If $G = 0$ there is nothing to prove.) Equating the terms in (1) that are homogeneous of degree $2k$ gives

$$(6) \quad c|x|^{2k} = \sum_{i=1}^N a_i \langle x, u_i \rangle^{2k}$$

where c is the coefficient of $|x|^{2k}$ in $G(x)$ and a_i is the coefficient of t^{2k} in $h_i(t)$. Applying the Laplacian to both sides of (6) and dividing by $2k(2k - 1)$ gives an equation of the same form with a new c , the same a_i 's and k replaced by $k - 1$. Repeating this process we see there are constants $c = c_k, \dots, c_0$ related by $c_{n-1} = c_n(2n + d - 2)/(2n - 1)$ such that

$$(7) \quad c_n|x|^{2n} = \sum_{i=1}^N a_i \langle x, u_i \rangle^{2n}$$

for $n = 0, \dots, k$. Note that $c_k \neq 0$ (because $2k = \text{deg } G$) and thus $c_0 \neq 0$. By (7), this means $a_1 + \dots + a_N \neq 0$.

Put $x = u$ in (7) and integrate over S_{d-1} . After dividing by ω_{d-1} and using (3) with $x = u_i$, the result is

$$(8) \quad c_n = \sum_{i=1}^N a_i \mu_n |u_i|^{2n} = (a_1 + \dots + a_N) \mu_n$$

for $n \leq k$. Using (3) and (7) with x replaced by a unit vector e , (8) becomes

$$(9) \quad \sum_{i=1}^N a_i \langle e, u_i \rangle^{2n} = (a_1 + \dots + a_n) \int_{S_{d-1}} \langle e, u \rangle^{2n} d\omega(u).$$

Taking $w_i = a_i/(a_1 + \dots + a_N)$ we see that (2) holds for $p(x) = \langle e, x \rangle^{2n}$, $n \leq k$. Since such functions span the space of even polynomials of degree $\leq 2k$, it follows by linearity that (2) holds for all p in that space. Thus the necessary constants w_i exist.

The extent to which the solution $h_i = w_i h_G$ of (1) is unique is made precise by

THEOREM 2. *If h is an even polynomial and $G(x) = \sum_{i=1}^N b_i h(\langle x, u_i \rangle)$ is radial, then $(b_1 + \dots + b_N)h = h_G$ where h_G is obtained from G as in Theorem 1. If in addition, G is not identically 0 then $b = b_1 + \dots + b_N$ is not 0 and the weights defined by $w_i = b_i/b$ make (2) valid for all even polynomials p of degree $\leq \text{deg } G$.*

PROOF. Equating the terms in $G(x) = \sum_{i=1}^N b_i h(\langle x, u_i \rangle)$ that are homogeneous of degree $2n$ gives

$$(10) \quad g_n|x|^{2n} = \sum_{i=1}^N b_i s_n \langle x, u_i \rangle^{2n}$$

where g_n is the coefficient of $|x|^{2n}$ in $G(x)$ and s_n is the coefficient of t^{2n} in $h(t)$. Take $x = u$ in (10) and integrate over S_{d-1} . After dividing by ω_{d-1} and using (3) with $x = u_i$, the result is

$$g_n = \sum_{i=1}^N b_i s_n \mu_n |u_i|^{2n} = (b_1 + \dots + b_N) s_n \mu_n$$

which proves that $bh = h_G$. If $G \neq 0$ then $h_G \neq 0$ so $b \neq 0$. Let $2k = \text{deg } G$. Taking $n = k$ in (10) gives (6) with $a_i = s_k b_i$. By the argument given for Theorem 1, the weights $w_i = a_i/(a_1 + \dots + a_N)$ make (2) valid. Since b_i/b gives the same weights, the proof is complete.

We close this paper by considering the distributional variant of (1) and by indicating how Theorem 1 generalizes. First we recall some notation and definitions.

If Ω is an open subset of R^d , $d \geq 1$, $\mathcal{D}(\Omega)$ denotes the space of C^∞ functions with compact support in Ω with the usual topology and $\mathcal{D}'(\Omega)$ denotes its dual, the space of distributions on Ω . If ϕ is in $\mathcal{D}(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, $\langle T, \phi \rangle_\Omega$ denotes T evaluated at ϕ . Set $I_1 = \{t \in R: -1 < t < 1\}$ and, again assuming $d \geq 2$, set $\Omega_1 = \{x \in R^d: |x| < 1\}$.

If ϕ is in $\mathcal{D}(\Omega_1)$ and h is in $\mathcal{D}'(I_1)$ take $Q_u\phi(t)$ to be the result of integrating ϕ over the hyperplane $\langle x, u \rangle = t$, notice that $Q_u\phi(t)$ is in $\mathcal{D}(I_1)$ as a function of t , call this function $Q_u\phi$, and define the distribution $1 \otimes_u h$ in $\mathcal{D}'(\Omega_1)$ by the formula

$$(11) \quad \langle 1 \otimes_u h, \phi \rangle_{\Omega_1} = \langle h, Q_u\phi \rangle_{I_1}.$$

Observe that if h is locally integrable on I_1 then $1 \otimes_u h$ is locally integrable on Ω and $1 \otimes_u h(x) = h(\langle x, u \rangle)$. Thus the distributional variant of (1) is

$$(12) \quad G = \sum_{i=1}^N 1 \otimes_{u_i} h_i$$

where G is in $\mathcal{D}'(\Omega_1)$ and h_i is in $\mathcal{D}'(I_1)$, $i = 1, \dots, N$.

To rule out obviously redundant terms in (1) and (12) we may assume

$$(13) \quad u_i \neq -u_j \quad \text{and} \quad u_i \neq u_j \quad \text{for } i \neq j.$$

THEOREM 3. *Let G be in $\mathcal{D}'(\Omega_1)$ and let h_1, \dots, h_N be in $\mathcal{D}'(I_1)$. If G is radial and (12) holds, then G must be a polynomial. If (13) also holds, then h_1, \dots, h_N must be polynomials.*

PROOF. Representing directional differentiation in the direction v by the usual notation D_v , we see that $Q_u D_v \phi = \langle u, v \rangle D Q_u \phi$. Hence if h is in $\mathcal{D}'(I_1)$, then $D_v(1 \otimes_u h) = \langle u, v \rangle (1 \otimes_u D h)$.

Choosing unit vectors v_1, \dots, v_N so that $\langle v_i, u_i \rangle = 0$ for $i = 1, \dots, N$ we see that $D_{v_1} \cdots D_{v_N} G = 0$. By Theorem 2 of [4] G must be a polynomial. Because of (13), we can arrange the choice of v_2, \dots, v_N so that $c_1 = \prod_{i=2}^N \langle v_i, u_1 \rangle \neq 0$. Applying D_{v_2}, \dots, D_{v_N} to both sides of (13) we have

$$(14) \quad D_{v_2} \cdots D_{v_N} G = c_1 (1 \otimes_{u_1} D^{N-1} h_1).$$

Since G is a polynomial, so is $F = D_{v_2} \cdots D_{v_N} G$. By (14) we see that $D_v F = 0$ for all v perpendicular to u_1 . Thus, for all $|x| < 1$, $F(x) = f(\langle x, u_1 \rangle)$ where $f(t) = F(tu_1)$. Using (14) it is easy to verify that $f = D^{N-1} h_1$. Hence h_1 is a polynomial. Since similar arguments apply to h_2, \dots, h_N , the proof is complete.

Combining Theorems 1 and 3 gives the following

COROLLARY. *If (2) holds for all even polynomials of degree $\leq v$, then every radial polynomial G of degree $\leq v$ enjoys a representation of the form (1). Conversely, if (12) holds where G is a radial distribution in $\mathcal{D}'(\Omega_1)$ and h_1, \dots, h_N are in $\mathcal{D}'(I_1)$, then G is a polynomial and there are constants w_1, \dots, w_N which make (2) valid for all even polynomials of degree $\leq \deg G$.*

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