EMBEDDED MINIMAL SURFACES IN 3-MANIFOLDS
WITH POSITIVE SCALAR CURVATURE

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Abstract. Let $M$ be a closed orientable Riemannian 3-manifold with positive scalar curvature. We prove that any embedded closed minimal surface in $M$ has a topological description as a generalized Heegaard surface. Also an existence theorem is proved which gives examples of such minimal surfaces.

0. Let $M^3$ be a closed orientable 3-manifold equipped with a Riemannian metric with positive scalar curvature. A closed surface $L \subset M$ is called minimal if the mean curvature of $L$ is zero everywhere. This is equivalent to the condition that, for all variations $L_t$ of $L$, the area $A_t$ of $L_t$ is stationary at $L$, i.e., $A'_0 = 0$.

Our aim is to give a simple topological description of such surfaces $L$ and to establish an existence result yielding some examples to illustrate the various cases that can arise. We now give a brief survey of previous work in this area.

We call $L \subset M$ a Heegaard surface if $M - L$ has 2 components whose closures are handlebodies. Heegaard surfaces $L, L' \subset M$ are said to be equivalent if there is a diffeomorphism $\phi: M \to M$ with $\phi(L) = L'$. Lawson [11] showed that if $M$ is a Riemannian $S^3$ with positive Ricci curvature then any embedded minimal surface is a Heegaard surface. Note also that Waldhausen [19] proved that any two Heegaard surfaces of the same genus in $S^3$ are equivalent. Lawson [10] gave examples of embedded minimal surfaces of every genus in $S^3$ with the standard metric. In addition, many such examples are constructed in [10] for other 3-manifolds with constant curvature 1, i.e. of the form $S^3/\Gamma$ where $\Gamma \subset SO(4)$ acts freely on $S^3$. Amongst these are pairs of embedded minimal surfaces in $S^3$ with the same genus for which there is no isometry of $S^3$ taking one onto the other.

More recently, Meeks-Simon-Yau (see §8 of [12]) established the same result as Lawson [11] for any $M^3$ with positive Ricci curvature. Also they obtained a topological characterization of orientable closed minimal surfaces embedded in $\#_{i=1}^{n} S^2 \times S^1$ equipped with a Riemannian metric with nonnegative scalar curvature (see §10 of [12]). We will give similar results for any $M$ with a metric with positive scalar curvature (or nonnegative scalar curvature if $M$ does not admit a flat metric) and will also treat nonorientable minimal surfaces.
1. In this section, we collect together some related results and definitions.

**Theorem 1** [6, 18]. Let $M$ be a closed orientable 3-manifold which admits a Riemannian metric with positive scalar curvature. Then $M$ can be written as a connected sum $M_1 \# \cdots \# M_k$, where for each $i$ either $M_i$ is a copy of $S^2 \times S^1$ or $\pi_1(M_i)$ is finite.

**Remark.** It is conjectured that every closed 3-manifold with finite fundamental group is diffeomorphic to a standard spherical 3-manifold with constant curvature 1. This has been proved by Hamilton [8] if the 3-manifold has a metric with positive Ricci curvature.

**Theorem 2** [5, 16]. Any 3-manifold $M$ which is a finite connected sum of copies of $S^2 \times S^1$ and of standard spherical 3-manifolds admits a Riemannian metric with positive scalar curvature.

We now state a result which is proved in [17] for closed 3-manifolds. The same argument clearly works in the case of compact 3-manifolds with suitable boundary.

**Theorem 3** [17]. Let $M$ be a compact 3-manifold with a Riemannian metric with positive scalar curvature and assume that $\partial M$ has nonnegative mean curvature with respect to the outward normal. Then $\pi_1(M)$ has no subgroups of the form $\pi_1(J)$, where $J$ is a closed orientable surface with genus $> 0$.

**Definition** (cf. [14]). A closed nonorientable surface $K$ embedded in a closed orientable 3-manifold $M$ is called a one-sided Heegaard surface if $M - K$ is an open handlebody.

**Remark.** If $K \subset M$ is a one-sided Heegaard surface then $M$ has a double covering $\tilde{M}$ so that the preimage of $K$ in $\tilde{M}$ is a Heegaard surface $\tilde{K} \subset \tilde{M}$ (cf. [14] and §2).

**Definition.** (1) Suppose $L^2 \subset M^3$ where $L$, $M$ are closed and orientable. $L$ is called a partial Heegaard surface for $M$ if one of the following holds:

(a) $L$ separates $M$ into 2 regions, each of which is a connected sum of an open handlebody with some closed 3-manifold.

(b) $M - L$ is the connected sum of 2 open handlebodies with a closed 3-manifold.

(2) If $K$ is a closed nonorientable surface then $K \subset M$ is a one-sided partial Heegaard surface in $M$ if $M - K$ is the connected sum of an open handlebody with a closed 3-manifold.

**Remarks.** By Kneser [9] and Milnor [13], any closed orientable $M^3$ can be uniquely expressed as a finite connected sum of prime 3-manifolds. (A closed orientable 3-manifold $Q$ is prime if either $Q = S^2 \times S^1$ or any embedded $S^2$ in $Q$ bounds a 3-cell.) Hence, if $L$ is a separating partial Heegaard surface, then $M$ has a prime factorization $M = M_1 \# \cdots \# M_{k+r+s}$ where $L$ can be viewed as a Heegaard surface for $M_1 \# \cdots \# M_k$, $M_{k+1} \# \cdots \# M_{k+r}$ lies on one side of $L$ and $M_{k+r+1} \# \cdots \# M_{k+r+s}$ is on the other side. Similarly, if $L$ is nonseparating, $M$ has a prime decomposition $M = M_1 \# \cdots \# M_{k+r} \# S^2 \times S^1$ where $L$ is a Heegaard surface for $M_1 \# \cdots \# M_k$ and the $S^2 \times S^1$ factor arises by forming a connected sum of the two open handlebodies which are the components of $M_1 \# \cdots \# M_k$ —
L. The other factors \( M_{k+1}, \ldots, M_{k+r} \) all miss \( L \). Finally if \( K \) is a one-sided partial Heegaard surface in \( M \), then \( M = M_1 \# \cdots \# M_{k+r} \) where each \( M_i \) is prime, \( K \) is a one-sided Heegaard surface for \( M_1 \# \cdots \# M_k \) and the other factors \( M_{k+i}, \) for \( 1 \leq i \leq r \), are all disjoint from \( K \).

**DEFINITION.** A closed surface \( L \subset M \) of genus \( > 0 \) is called *incompressible* if there is no disk \( D \subset M \) with \( D \cap L = \partial D \) a noncontractible loop on \( L \). A 2-sphere \( S \subset M \) is incompressible if \( S \) does not bound a 3-cell in \( M \).

**2.** Suppose \( K \) is a closed nonorientable surface embedded in a closed orientable 3-manifold \( M \). We will construct various double coverings \( \tilde{M} \) of \( M \) for which the preimage \( \tilde{K} \) of \( K \) in \( \tilde{M} \) is orientable. Let \( M = N(K) \cup Y \), where \( N(K) \) is a small closed regular neighbourhood of \( K \) and \( Y = M - \text{int} N(K) \). Let \( \tilde{K} \) be the orientable double covering of \( K \) and let \( \tilde{Y} \) be any double covering of \( Y \) such that \( \partial \tilde{Y} \) is disconnected. We allow the possibility that \( \tilde{Y} \) itself is disconnected and, in this case, \( \tilde{Y} \) is the disjoint union of two copies of \( Y \). Note that a connected double covering \( \tilde{Y} \) of \( Y \) exists if and only if the map \( H_1(K; \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2) \) induced by inclusion is not onto. There is a double covering of \( N(K) \) by \( \tilde{K} \times [-1,1] \) with the preimage of \( K \) equal to \( \tilde{K} \times \{0\} \), since \( N(K) \) is a twisted line bundle over \( K \).

The double covering \( \tilde{M} \) is obtained by gluing \( \tilde{K} \times [-1,1] \) to \( \tilde{Y} \) by a diffeomorphism \( \psi: \tilde{K} \times \{0\} \to \partial \tilde{Y} \). \( \psi \) is chosen so that the double covering projections \( \tilde{K} \times [-1,1] \to N(K) \) and \( \tilde{Y} \to Y \) match up. So we obtain \( \tilde{M} = \tilde{K} \times [-1,1] \cup \psi \tilde{Y} \) and there is a double covering \( p: \tilde{M} \to M \) with \( p^{-1}(K) = \tilde{K} \times \{0\} \).

**3. THEOREM 4.** Suppose \( M \) is a closed orientable Riemannian 3-manifold. Assume that the scalar curvature is positive, or if \( M \) admits no flat metric then it suffices to suppose that the scalar curvature is nonnegative. If \( L \subset M \) is minimal then \( L \) is a partial Heegaard surface for \( M \).

**PROOF.** This will follow by applying Theorem 3 to \( M \) split along \( L \). Firstly, if \( L \) is orientable and nonseparating, we obtain a connected compact manifold \( M_1 \) with two copies of \( L \) in \( \partial M_1 \), by dividing \( M \) along \( L \). If \( L \) separates \( M \), two compact manifolds \( M_2 \) and \( M_3 \) are constructed with \( \partial M_2 \) and \( \partial M_3 \) both a copy of \( L \), by splitting \( M \) along \( L \). Finally, if \( L \) is nonorientable, let \( p: \tilde{M} \to M \) be the double covering constructed in §2, where \( \tilde{M} = \tilde{L} \times [-1,1] \cup \psi \tilde{Y} \) and \( \tilde{Y} \) is disconnected. Let \( Y_1, Y_2 \) be the components of \( \tilde{Y} \) with \( \partial Y_1 = \psi(\tilde{L} \times \{-1\}) \). The compact manifold \( M_4 \subset \tilde{M} \) given by \( M_4 = \tilde{L} \times [-1,0] \cup Y_1 \) satisfies \( \partial M_4 \), is a copy of \( \tilde{L} \) and \( p \) maps int \( M_4 \) diffeomorphically onto \( M - L \).

By Theorem 3, there are no subgroups of \( \pi_1(M_i) \) of the form \( \pi_1(J) \), where \( J \) is a closed orientable surface of positive genus and \( 1 \leq i \leq 4 \). Note that if \( M \) is assumed only to have a metric of nonnegative scalar curvature, then as in [17] the metric can be approximated by one with positive scalar curvature (since \( M \) admits no flat metric) and Theorem 3 applies.

To complete the proof we apply Dehn's lemma and the Loop theorem to the surfaces \( \partial M_i \), for \( 1 \leq i \leq 4 \). Suppose \( G \) is a component of \( \partial M_i \) and \( \{ D_u: 1 \leq u \leq v \} \) is a maximal family of disjoint compressing disks for \( G \), i.e. \( \partial D_u \subset G \), \( \text{int} D_u \subset \text{int} M_i \),
\( \partial D_u \) is noncontractible in \( G \) for all \( u \) and no two curves \( \partial D_u \) and \( \partial D_w \) are parallel on \( G \), for \( u \neq w \). Let \( N = N(\bigcup_u D_u \cup G) \) be a small closed regular neighbourhood in \( M_i \). If some component \( J \) of \( \partial N - G \) has genus \( > 0 \), then since \( \pi_1(J) \subset \pi_1(M_i) \) (by Dehn’s lemma and the Loop theorem) we get a contradiction to Theorem 3. So \( \partial N \) consists of \( G \) together with 2-spheres. It follows immediately that \( M_3, M_4 \) and \( M_4 \) are all connected sums of a handlebody \( N \) with cells attached along the 2-spheres in \( \partial N \) with a closed 3-manifold. Similarly, \( M_1 \) is a connected sum of two handlebodies with a closed 3-manifold, and so \( L \) is a partial Heegaard surface in all cases.

**Remarks.** (1) In the case that \( M^3 = \#_{i=1}^n S^2 \times S^1 \), Theorem 4 gives the results in §10 of [12]. By results of Haken [7] and Waldhausen [19], any two orientable partial Heegaard surfaces of the same genus in \( \#_{i=1}^n S^2 \times S^1 \) are equivalent if they are both separating or both nonseparating and if there are the same numbers of \( S^2 \times S^1 \) factors in the components of the complements of the surfaces (cf. [12, §10] also).

(2) Engmann [4] and Birman [1] have given examples of two Heegaard surfaces of genus 2 in a connected sum of lens spaces \( L(p, q) \# L(p', q') \) which are not equivalent. On the other hand, Bonahon and Otal [3] have recently shown that any two Heegaard surfaces with the same genus in any lens space \( L(p, q) \) are equivalent.

4. **Theorem 5.** Suppose \( M \) is a closed orientable Riemannian 3-manifold with no orientable incompressible surfaces of genus \( > 0 \). If \( H_1(M, Z) \) has 2-torsion then \( M \) has an embedded minimal nonorientable surface which is a one-sided partial Heegaard surface. Consequently, there are double covers \( \tilde{M} \) of \( M \) with orientable minimal partial Heegaard surfaces.

**Proof.** Since \( H_1(M, Z) \) has 2-torsion, we can choose a torsion element \( \alpha \in H_1(M, Z) \) such that \( \alpha \otimes 1 \neq 0 \) in \( H_1(M, Z) \otimes Z_2 \). Exactly as in [14], a nonorientable incompressible surface \( K \subset M \) can be found so that the intersection number mod 2 of the class of \( K \) in \( H_2(M, Z_2) \) and \( \alpha \) is one. Moreover, \( K \) is a one-sided partial Heegaard surface for \( M \), because there are no orientable incompressible surfaces in \( M \) other than 2-spheres. (In [14] it is proved that \( K \) is a one-sided Heegaard surface if \( M \) is irreducible. Clearly the same argument works here.) By [12] there is a stable minimal incompressible surface \( K' \subset M \), with \( K' \) homeomorphic to \( K \). \( K' \) is also a partial Heegaard surface in \( M \). The preimage \( \tilde{K} \) of \( K' \) in a double cover \( \tilde{M} \) of \( M \), as constructed in §2, is then an orientable minimal partial Heegaard surface for \( \tilde{M} \). This completes the proof.

**Examples.** (1) The above hypotheses are satisfied by a finite connected sum \( M = M_1 \# \cdots \# M_k \), where each \( M_i \) is either a copy of \( S^2 \times S^1 \) or \( \pi_1(M_i) \) is finite and at least one \( \pi_1(M_i) \) is even, solvable and not generalised binary tetrahedral by cyclic.

(2) Let \( M = L(4,1) \# S^2 \times S^1 \) with any Riemannian metric. Now \( L(4,1) \) has an incompressible Klein bottle \( K \) (see [14]) which can be minimally embedded in \( M \), as in Theorem 5. \( M \) has two double coverings, \( \tilde{M}_1 = RP^3 \# S^2 \times S^1 \) and \( \tilde{M}_2 = RP^3 \# S^2 \times S^1 \# S^2 \times S^1 \), in which \( K \) is covered by a minimal torus \( \tilde{K} \) (cf. §2). In the former, the torus is nonseparating, while in the latter case, it is a separating surface. So we obtain all three types of partial Heegaard surfaces realized by minimal surfaces.
All the lens spaces \( L(2p, q) \) have embedded incompressible nonorientable surfaces \( K \) (cf. [2 and 14] for properties of these surfaces). As in Theorem 5, \( K \) can be chosen to be minimal and lifts to minimal Heegaard surfaces in both \( L(p, q) \) and \( S^3 \). For example, with the standard spherical metric on \( L(2p, q) \) this produces many new embedded minimal surfaces in \( S^3 \), as compared with Lawson’s constructions in [10]. See also [15] for a related procedure for finding minimal surfaces, based on an equivariant version of [12].

Note added in proof. Similar results have been obtained by S. Almeida, Ph. D. Thesis, State University of New York, Stony Brook (December 1982).

References

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