CONTINUOUS FUNCTIONS ON THE SPACE OF PROBABILITIES

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Abstract. Weiss and Dubins discovered that any continuous function \( g(P) \) on the space of probabilities \( \mathcal{P} \) of a compact Hausdorff space \( K \) is of the form \( \int f \, dP \) for some continuous function \( f \) on \( K^\infty \). A short proof is given here.

Let \( \mathcal{P} \) be the space of probabilities on the Borel \( \sigma \)-field of the unit interval \( K = [0, 1] \) equipped with the weak* topology so that \( \mathcal{P} \) is again compact. It is easy to see that if \( f \) is a real continuous function on \( K^\infty \) then \( g(P) = \int f \, dP \) is a continuous function on \( \mathcal{P} \). Recently B. Weiss and L. E. Dubins [1] proved the converse. Namely,

**Theorem.** Given a real continuous function \( g \) on \( \mathcal{P} \) there is a real continuous function \( f \) on \( K^\infty \) such that \( g(P) = \int f \, dP \) for all \( P \) in \( \mathcal{P} \).

For \( f \in C(K^\infty) \), let the function \( g \) defined above be denoted by \( Tf \). \( T \) is obviously a continuous linear operator on \( C(K^\infty) \) into \( C(\mathcal{P}) \). Observe that if \( f_1, f_2 \) are in \( C(K^\infty) \) then \( f \) defined on \( K^\infty \) by

\[
  f(x_1, x_2, \ldots) = f_1(x_1, x_3, \ldots) f_2(x_2, x_4, \ldots)
\]
is continuous and \( Tf = Tf_1 \cdot Tf_2 \). As a consequence range of \( T \) is a subalgebra of \( C(\mathcal{P}) \) containing constants and it clearly separates points. We shall now show that \( T^* \) has closed range. This, by Banach’s closed range theorem [2] implies that range of \( T \) is closed and, hence, by the Stone-Weierstrass theorem, must be all of \( C(\mathcal{P}) \) as claimed.

Observe that \( T^*: \mathcal{M}(\mathcal{P}) \to \mathcal{M}(K^\infty) \) is given by

\[
  T^* \mu(A) = \int P^\infty(A) \, d\mu(P) \quad \text{for } \mu \in \mathcal{M}(\mathcal{P}).
\]

\( \mathcal{M}(X) \) is the space of finite signed measures on \( X \) with total variation norm. As \( T \) has dense range, \( T^* \) is one-to-one. If \( \mu \) is positive then so is \( T^* \mu \). If \( \mu_1, \mu_2 \) are positive and orthogonal then so are \( T^* \mu_1 \) and \( T^* \mu_2 \). Indeed, if \( L \subset \mathcal{P} \) is a Borel set such that \( \mu_1 \) sits on \( L \) and \( \mu_2 \) sits on \( L^c \), then a simple application of the Glivenko-Cantelli Lemma implies that \( T^* \mu_1 \) sits on \( A \) and \( T^* \mu_2 \) sits on \( A^c \) where

\[
  A = \left\{ (x_1, x_2, \ldots) \in K^\infty: \frac{1}{n} \sum_{1}^{n} \delta_{x_i} \text{ has a limit in } \mathcal{P} \text{ and that limit } \in L \right\}.
\]
As usual $\delta_x$ is the point mass at $x$. This implies that if $\mu = \mu^+ - \mu^-$ is the canonical Jordan Hahn decomposition of $\mu \in M(\mathcal{P})$ then $T^*\mu^+ - T^*\mu^-$ is the canonical decomposition of $T^*\mu$. This in turn implies that the total variations of $\mu$ and $T^*\mu$ are equal. In other words, $T^*$ is a norm preserving map and hence $\text{Range } T^*$ is closed. This completes the proof.

**Remark 1.** The same proof applies if $K$ is any compact Hausdorff space and $\mathcal{P}$ the probabilities on the Baire $\sigma$ field. This is the form proved in [1].

**Remark 2.** Dr. G. Jogesh Babu has yet another proof of the Theorem patterned after the original proof in [1] but more probabilistic in nature.

**Remark 3.** The representation theorem of Hewitt-Savage for symmetric probabilities identifies range of $T^*$ as precisely the linear span of symmetric probabilities.

**References**


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