PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 3, November 1985

OPEN AND MONOTONE FIXED POINT FREE MAPS ON UNIQUELY ARCWISE CONNECTED CONTINUA

LEE MOHLER AND LEX G. OVERSTEEGEN[1]

ABSTRACT. In this note we will construct uniquely arcwise connected continua admitting open and monotone fixed point free mappings, respectively. We will also show that each locally one-to-one map on a uniquely arcwise connected continuum has a fixed point.

Introduction. A *continuum* is a compact and connected metric space. A continuum X is said to be *uniquely arcwise connected* (u.a.c) if given any two points $x, y \in X$, $x \neq y$, there is a unique arc in X whose endpoints are x and y. This is equivalent to saying that X is arcwise connected and contains no simple closed curves. It is known **[5]** that u.a.c. need not have the fixed point property for continuous mappings but **[3]** they do have the fixed point property for homeomorphisms.

A continuum X is called *decomposable* provided there exist two proper subcontinua $A, B \subset X$ such that $X = A \cup B$ and *hereditarily decomposable* provided each subcontinuum is decomposable. A continuum is called *indecomposable* provided it is not decomposable. A map $f\colon X \to Y$ is called *open* provided $f(U)$ is open for each open set $U \subset X$ and *monotone* provided $f^{-1}(y)$ is connected for each $y \in Y$. A map $f\colon X \to Y$ is *open on $A \subset X$* provided, for each $x \in A$, there exists an open set U containing x such that $f(U)$ is open. A map $f\colon X \to Y$ is called *locally one-to-one* provided that for each $x \in X$ there exists an open set $U \subset X$ containing x such that $f|U$ is one-to-one. We will show that each locally one-to-one map $f\colon X \to X$ where X is a u.a.c. continuum has a fixed point. We will also show that this result is the best possible by constructing hereditarily decomposable u.a.c. continua admitting monotone and open fixed point free maps, respectively. It follows from a result of Hagopian **[2]** that these examples are nonplanar.

If x and y are points in a u.a.c. continuum X, we denote by $[x, y]$ the unique irreducible arc joining them. Let X be a continuum and let $x \in X$. By $o(x, X)$ we denote the minimal cardinal number α with the property: For each open set $U \subset X$ containing x, there exists an open set V, $x \in V \subset U$, such that $|\mathrm{Bd}(V)| \leqslant \alpha$. We let ω denote the first infinite cardinal number. We will make use of the following observation:

LEMMA 0. *Let Z be an indecomposable subcontinuum of a continuum X and let $z \in Z$. Then $o(z, X) > \omega$.*

Received by the editors April 20, 1984 and, in revised form, December 11, 1984.

The results of this paper were announced at the AMS Summer Meeting in Bellingham, Washington, 1982.

1980 *Mathematics Subject Classification*. Primary 54F20; Secondary 54F50.

[1] The second author was supported in part by NSF grant number MCS-8104866.

0002-9939/85 $1.00 + $.25 per page

THEOREM 1. *Let X be a u.a.c. continuum and let $f\colon X \to X$ be a locally one-to-one map. Then f has a fixed point.*

PROOF. Suppose $Y_0 = f(X) \not\subseteq X$. Define Y_α by transfinite induction as follows: $Y_{\alpha+1} = f(Y_\alpha)$ and $Y_\gamma = \bigcap_{\alpha<\gamma} Y_\alpha$ if γ is a limit ordinal. Then $Y_\alpha \subset Y_\beta$ if $\alpha < \beta$. It follows easily (using transfinite induction) that Y_α is uniquely arcwise connected for each α. Since X is a continuum, there exists an index α_0 such that $Y_{\alpha_0} = Y_{\alpha_0+1}$. Then $f|Y_{\alpha_0}\colon Y_{\alpha_0} \twoheadrightarrow Y_{\alpha_0}$ is a locally one-to-one map from the u.a.c. continuum Y_{α_0} onto Y_{α_0}. We claim that f is one-to-one. To this end, suppose there exist $x_1 \neq x_2 \in X$ such that $f(x_1) = f(x_2)$. Let A be an arc joining x_1 and x_2, let $<$ be an order on A and let 0 denote the minimum of A. Let $z_1 = \min\{a \in A \mid f(a) \in f([0, a))\}$, and let $z_0 = \max\{a \in [0, z_1) \mid f(a) = f(z_1)\}$. Then $z_0 < z_1$ and $f([z_0, z_1])$ is a simple closed curve. This contradicts the fact that X is uniquely arcwise connected. Hence, $f|Y_{\alpha_0}\colon Y_{\alpha_0} \twoheadrightarrow Y_{\alpha_0}$ is a homeomorphism. It follows from [3] that $f|Y_{\alpha_0}$ has a fixed point and the proof of the theorem is complete.

Recall that Young's example [5] of a fixed point free map on a uniquely arcwise connected continuum X consists of a double Warsaw circle W (see Figure 1) in the xy-plane in $\mathbf{R}^3$, an arc joining the points $a = (0, -1, 0)$ and $b = (0, 1, 0)$ and a ray R with initial point $v = (0, 0, 0)$ which compactifies onto W.

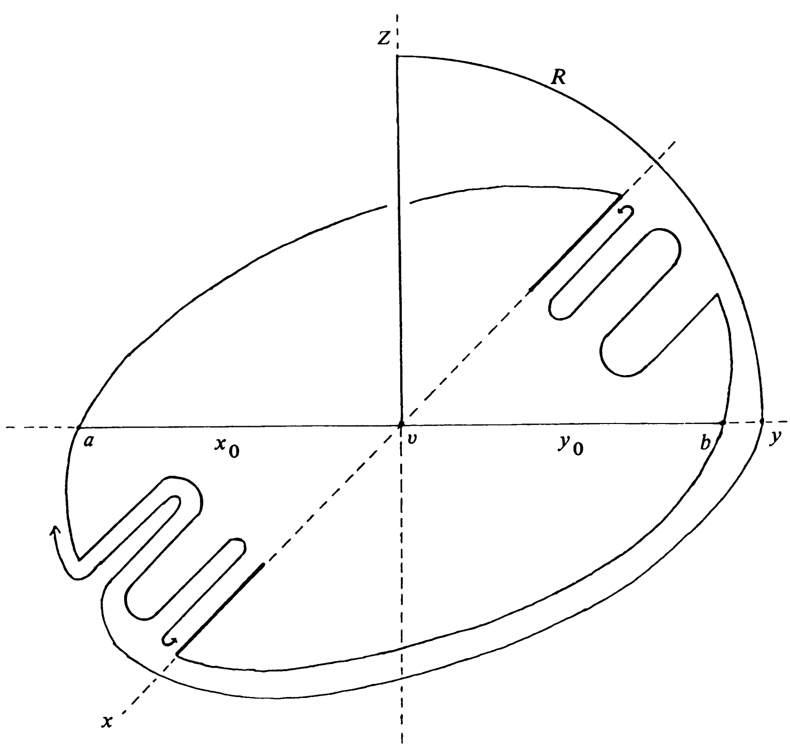

FIGURE 1

The fixed point free map $f\colon X \twoheadrightarrow X$ rotates $W = 180°$, interchanging the points a and b, maps $[a, x_0]$ onto $[b, v]$ and $[b, y_0]$ onto $[a, v]$ and maps the ray R into itself such that $f|R$, $f|[x_0, v]$ and $f|[y_0, v]$ are one-to-one. It follows that f is open on $X \setminus \{x_0, y_0\}$ and $f|X \setminus [x_0, y_0]$ is one-to-one (and, in particular, monotone). We will assume that $f(0, y, 0) = (0, -1 - 2y, 0)$ for each $y \in [a, x_0]$ and $f(0, y, 0) = (0, 1 - 2y, 0)$ for each $y \in [y_0, b]$.

EXAMPLE 1. There exists a hereditarily decomposable u.a.c. continuum Z admitting a monotone fixed point free map $g\colon Z \twoheadrightarrow Z$.

PROOF. Let X denote Young's example and let $f\colon X \to X$ be the fixed point free map described above. Note that $f|X \setminus [x_0, y_0]$ is one-to-one and hence monotone. We will connect the points $(0, y, 0)$ and $(0, -y, 0)$ on $[a, b]$ with a continuum such that f admits a monotone fixed point free extension. Let $C \subset I$ be the standard Cantor ternary set in $[0, 1]$ obtained by removing the "middle third" intervals. Let $h\colon I \twoheadrightarrow I$ be a monotone map such that $h(C) = I$ and $h|C$ is the "standard" Lebesgue singular function (i.e. $h(\frac{1}{3}) = h(\frac{2}{3}) = \frac{1}{2}$, $h(\frac{1}{9}) = h(\frac{2}{9}) = \frac{1}{4}, \ldots$). Let W be the double Warsaw circle described above and consider $W \times C$. Let $(W \times C)^* = (W \times C)/(W \times \{1\})$ and let $\phi\colon W \times \{0\} \to W \subset X$ denote the natural homeomorphism. Let Z be obtained from the disjoint union of $(W \times C)^*$ and X by identifying each point of $(w, 0)$ with $\phi((w, 0))$, each point (a, c) with $(0, h(c) - 1, 0) \in X$ and each point (b, c) with $(0, 1 - h(c), 0)$. Then Z is a u.a.c. continuum. Each pair of points $(0, y, 0)$ and $(0, -y, 0)$ where y is *not* of the form $k/2^n$ are joined by a homeomorphic copy W_y of W and each pair of points $(0, y, 0)$ and $(0, -y, 0)$ where y is of the form $k/2^n$ are joined by a continuum W_y which is obtained from two homeomorphic copies of W by identifying the points corresponding to a and b, respectively. Define the fixed point free extension $g\colon Z \to Z$ such that $g(W_y) = f(0, y, 0) \in R$ if $|y| < \frac{1}{2}$, $g|W_y\colon W_y \twoheadrightarrow W_{-2y-1}$, $-1 \leqslant y \leqslant -\frac{1}{2}$, and $g|W_y\colon W_y \to W_{-2y+1}$, $\frac{1}{2} \leqslant y \leqslant 1$, are the natural homeomorphism followed by a $180°$ rotation. It is not difficult to see that g is the required fixed point free extension of f. Note that if $x \in X \setminus [a, b] \subset Z$, then $o(x, X) = \omega$. It is not difficult to see that $[a, b] \cup Z \setminus X$ is hereditarily decomposable. By Lemma 0, Z is hereditarily decomposable.

We will next construct a hereditarily decomposable u.a.c. X admitting an open fixed point free map. We start out with some preliminaries. By a *λ-dendroid D* we mean a hereditarily decomposable, hereditarily unicoherent continuum. Note that for each pair of points in D, there is at most one arc joining them. A *dendroid* is an arcwise connected λ-dendroid.

EXAMPLE 2. There exists a λ-dendroid D admitting an open and monotone map ϕ: $D \twoheadrightarrow [0, 1]$ such that $\phi^{-1}(x)$ is a point for each $x \in [0, \frac{3}{4}] \cup [\frac{7}{8}, 1]$ and D contains exactly two arc components A_1 and A_2 such that $f^{-1}(0) \in A_1$ and $f^{-1}(1) \in A_2$.

PROOF. In [4], the second author constructed a dendroid Z admitting a monotone and open retraction $r\colon Z \twoheadrightarrow J = [0, 1]$ where J is an arc such that $r^{-1}(t)$ is nondegenerate for each $t \in J$. Let $Z^* = Z/r^{-1}(0)$, let $\pi\colon Z \to Z^*$ be the natural projection and let $r^*\colon Z^* \to [0, 1]$ be defined by $r^*(z) = r(z)$ if $z \in Z \setminus r^{-1}(0) \subset Z^*$ and $r^*(\pi \circ \{r^{-1}(0)\}) = 0$. Then r^* is an open and monotone retraction of the dendroid Z^* onto $[0, 1]$ such that $r^{*-1}(t)$ is nondegenerate for each $t \in (0, 1]$. Let

$Z^- = r^{*-1}([0,1))$, $J^* = \pi(J)$, let $J^- = J^* \setminus \{1\} \subset Z^-$ and let B be a homeomorphic copy of $r^{*-1}(1) \subset Z^*$. Then J^- is a ray. Let Z^+ be a compactification of Z^- with remainder B such that $Z^- \cap B = \varnothing$, $\mathrm{Cl}(J^-) \setminus J^- = B$. Moreover, since $r^{*-1}(t)$ is nondegenerate (and has diameter greater than some number $\varepsilon > 0$), we can choose the compactification such that $\operatorname{Lim} r^{*-1}(t_n) = B$ for each sequence $t_n \in J^-$ such that $\lim(t_n) = \{1\}$. It follows that there exists an open and monotone map $r^+ \colon Z^+ \twoheadrightarrow [0,1]$ defined by $r^+(z) = r^*(z)$ if $z \in Z^- \subset Z^+$ and $r^+(z) = 1$ if $z \in B$. Moreover, Z^+ is a λ-dendroid which contains exactly two arc components, namely Z^- and B. Let D be the λ-dendroid obtained from the disjoint union of Z^*, Z^+ and the arcs $[-2,-1]$ and $[1,2]$ by identifying $r^{*-1}(1) \subset Z^*$ with $B \subset Z^+$ under the natural homeomorphism, the point $(r^+)^{-1}(0)$ with -1 and the point $r^{*-1}(0)$ with 1. It is now not difficult to construct the required map $\phi \colon D \twoheadrightarrow [0,1]$. We will picture the λ-dendroid D as in Figure 2 and label the point $\phi^{-1}(0)$ and $\phi^{-1}(1)$, a and b, respectively.

It is not difficult to see that the "double Warsaw" circle in Young's example can be replaced by any u.a.c. containing exactly two arc components which admits a 180° rotation. We will replace (see Figure 3) W by the disjoint union of two homeomorphic copies, D^+ and D^- of D where the point corresponding to a (b) in D^+ is identified with the point corresponding to b (a, respectively) in D_-. We will label the first point by α and the second by β.

Let X_1 denote this modified Young example and let $f_1 \colon X_1 \to X_1$ be the corresponding fixed point free map. As indicated above, f_1 is open on $X \setminus f_1^{-1}(v)$. Put $f_1^{-1}(v) = V_1 = \{x_0, y_0\}$.

The required u.a.c. continuum X will be constructed as an inverse limit. Let $X = \varprojlim(X_n, r_m^n)$ where each map $r_m^n \colon X_n \to X_m$ is a retraction. We will assume that $X_m \subset X_n \subset X$ for each $n \leqslant m$ and denote by $r_n \colon X \to X_n$ the natural projection. Note that r_n is a retraction.

Lemma 2. *Let $X = \varprojlim(X_n, r_m^n)$, where $r_m^n \colon X_n \to X_m$ is a retraction and X_n is a u.a.c. continuum for each $n \geqslant 1$. Suppose that for each $x \in X$, there exists $n_0 \geqslant 1$ such that $r_n(x) = r_{n_0}(x)$ for all $n \geqslant n_0$. Then x is a u.a.c. continuum and X is homeomorphic with $\bigcup_n X_n$. We will identify X with $\bigcup_n X_n$. Hence for each $x \in X$ there exists n such that $r_n(x) = x$.*

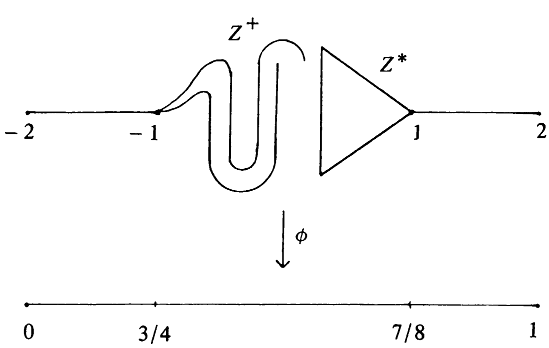

FIGURE 2

PROOF. Suppose $x, y \in X$. Let n be so large that $x = r_n(x)$ and $y = r_n(y)$. Since X_n is u.a.c., there exists an arc $J \subset X_n \subset X$ joining x and y. Hence X is arcwise connected. Suppose next that $S \subset X$ is a simple closed curve. Then there exists n such that $r_n(S) \subset X_n$ contains a simple closed curve. This contradicts the fact that X_n is u.a.c. and completes the proof.

If X is a continuum and $a, b \in A \subset X$, we say that Y is obtained from X by doubling $A \setminus \{a, b\}$ provided Y is homeomorphic to the space obtained from the disjoint union of X and a disjoint copy of A by identifying the points corresponding to a and b, respectively. We will always assume that $X \subset Y$. Hence there exists a natural retraction $r: Y \twoheadrightarrow X$.

EXAMPLE 3. There exists a hereditarily decomposable uniquely arcwise connected continuum X admitting an open fixed point free map $f: X \twoheadrightarrow X$.

PROOF. Let X_1 be the modified Young example (see Figure 3) constructed above and let $f_1: X_1 \to X_1$ denote the corresponding fixed point free map. Put $V_1 = f_1^{-1}(v) = \{x_0, y_0\}$ ($x_0 = (0, -\frac{1}{2}, 0)$, $y_0 = (0, \frac{1}{2}, 0) \in \mathbf{R}^3$). Then f_1 is open on $X_1 \setminus \{x_0, y_0\}$. Recall that there exists an open map $\phi: D \to [0,1]$ (see Example 2). Let X_2 (see Figure 4) be the space obtained from X_1 by doubling $[\alpha, x_0] \cup D^- \setminus \{x_0, \beta\}$ and $[y_0, \beta] \cup D^+ \setminus \{y_0, \alpha\}$.

Let $r_1^2: X_2 \twoheadrightarrow X_1$ be the natural retraction and let $f_2: X_2 \to X_1$ be an extension of f_1 such that f_2 maps the attached copy of $[x_0, \alpha] \cup D^-$ ($[y_0, \beta] \cup D^+$) open onto $[\beta, v]$ ($[\alpha, v]$, respectively) such that $f_2^{-1}(t)$ consists of exactly two points for each $t \in [\alpha, x_0] \cup [\beta, y_0]$. Note that f_2 is open on the set $X_2 \setminus f_2^{-1}(\{x_0, y_0\}) \cup \{\alpha, \beta\}$ and that $r_1^2 \circ f_2: X_2 \twoheadrightarrow X_1$ is open. Put $f_2^{-1}(x_0) = \{y_1, y_2\} = A_1$ and $f_2^{-1}(y_0) = \{x_1, x_2\} = B_1$. Let X_3 be the space obtained from X_2 by doubling $[x_1, \alpha] \cup D^- \setminus \{x_1, \beta\}$, $[x_2, \alpha] \cup D^- \setminus \{x_2, \beta\}$, $[y_1, \beta] \cup D^+ \setminus \{y_1, \alpha\}$ and $[y_2, \beta] \cup D^+ \setminus \{y_2, \alpha\}$ and let $r_2^3: X_3 \to X_2$ be the natural retraction. Let $f_3: X_3 \to X_3$ be an extension of f_2 such that f_3 restricted to the closure of each component of $X_3 \setminus X_2$ is one-to-one. Put

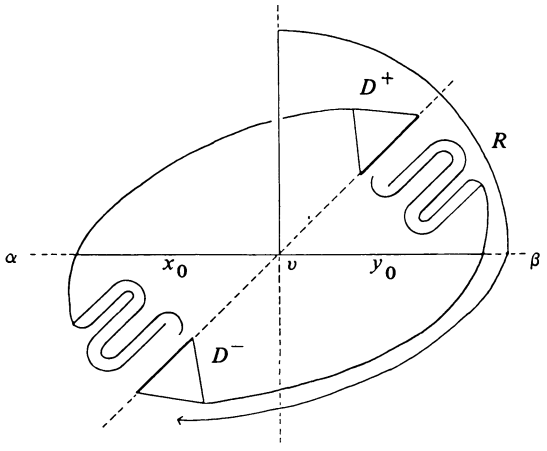

FIGURE 3

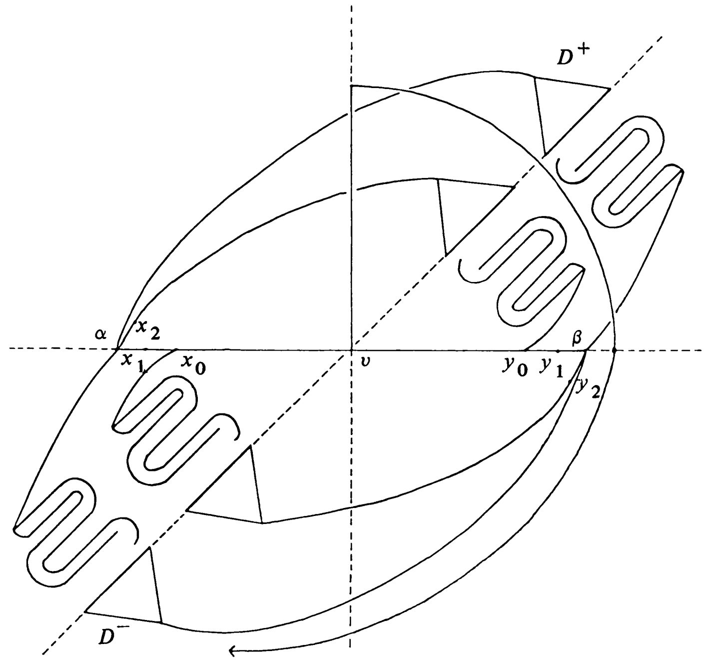

FIGURE 4

$V_3 = f_3^{-1} \circ f_3^{-1} \circ f_3^{-1}(v) = f_3^{-1}\{x_1, x_2, y_1, y_2\}$, then f_3 is open on the set $X \setminus V_3 \cup \{\alpha, \beta\}$ and $r_2^3 \circ f_3$: $X_3 \to X_2$ is open. Note that, for each point $x \in V_3$, $f_3^{-1}(x)$ consists of exactly two points. Put $A_2 = f_3^{-1}(A_1)$ and $B_2 = f_3^{-1}(B_1)$.

Suppose X_n, f_n, r_m^n, A_{n-1} and B_{n-1} have been constructed for $m \leqslant n \leqslant n_0$ such that f_{n_0} is open on the set $X_{n_0} \setminus A_{n-1} \cup B_{n-1} \cup \{\alpha, \beta\}$. Let X_{n_0+1} be obtained from X_{n_0} by doubling $[\alpha, a] \cup D^- \setminus \{\alpha, \beta\}$ for each $a \in A_{n-1}$ and $[b, \beta] \cup D^+ \setminus \{b, \alpha\}$ for each $b \in B_{n-1}$ and let $r_{n_0}^{n_0+1}$: $X_{n_0+1} \to X_{n_0}$ be the natural projection. Let f_{n_0+1}: $X_{n_0+1} \to X_{n_0+1}$ be an extension of f_{n_0} such that f_{n_0+1} restricted to the closure of each component of $X_{n_0+1} \setminus X_{n_0}$ is one-to-one. If C is the closure of a component of $X_{n_0+1} \setminus X_{n_0}$, then $f_{n_0+1}(C) = C'$ is the closure of a component of $X_{n_0} \setminus X_{n_0-1}$. We will define f_{n_0+1} such that

$$r_{n_0-1}^{n_0+1} \circ f_{n_0+1}\big|C = r_{n_0-1}^{n_0} \circ f_{n_0} \circ r_{n_0}^{n_0+1}\big|C. \tag{1}$$

Put $A_{n_0} = f_{n_0+1}^{-1}(A_{n_0-1})$ and $B_{n_0} = f_{n_0+1}^{-1}(B_{n_0-1})$. Then f_{n_0+1} is open on the set $X_{n_0+1} \setminus A_{n_0} \cup B_{n_0} \cup \{\alpha, \beta\}$ and $r_{n_0}^{n_0+1} \circ f_{n_0+1}$: $X_{n_0+1} \to X_{n_0}$ is open. Let $X = \varprojlim(X_n, r_m^n)$. By Lemma 2, X is a u.a.c. continuum. It follows from (1) that the sequence $r_{n-1}^n \circ f_n$: $X_n \to X_{n-1}$ induces an open map f: $X \twoheadrightarrow X$. It is not difficult to check that f is the required fixed point free map. It is not difficult to see that if $Q = \{x \in X | o(x, X) \leqslant \omega\}$ (note that $\alpha, \beta \in Q$) and if $Z \subset X \setminus Q$ is a continuum, then Z is a subset of a homeomorphic copy of the λ-dendroid D constructed in Example 2. (Recall that the retractions r_n^{n+1} used in the inverse limit description of X, are "nondegenerate" essentially only on the sets $D^- \cup D^+$.) By Lemma 0, X is hereditarily decomposable.

REMARK. A. Conner has announced the existence of an open and monotone fixed point free map on a u.a.c. continuum. This example is obtained from the hyperspace of subcontinua $C(P)$ of the pseudo-circle P to which a ray R is added starting at the point $\{P\} \in C(P)$ which compactifies on the "base" $P \subset C(P)$. The following problem remains open.

Problem 1. Does there exist an hereditarily decomposable u.a.c. continuum which admits an open and monotone fixed point free map?

REFERENCES

1. A. Conner (to appear).

2. C. L. Hagopian, *Uniquely arcwise connected plane continua have the fixed point property*, Trans. Amer. Math. Soc. **248** (1979), 85–104.

3. L. Mohler, *The fixed point property for homeomorphisms of* 1*-arcwise connected continua*, Proc. Amer. Math. Soc. **25** (1975), 451–456.

4. L. G. Oversteegen, *Open retractions and locally confluent mappings of certain continua*, Houston J. Math. **6** (1980), 113–125.

5. G. G. Young, Jr., *Fixed point theorems for arcwise connected continua*, Proc. Amer. Math. Soc. **11** (1960), 880–884.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA IN BIRMINGHAM, BIRMINGHAM, ALABAMA 35294