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INVERSE SYSTEMS OF ABSOLUTE RETRACTS AND ALMOST CONTINUITY

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ABSTRACT. Suppose that Y is the inverse limit of a sequence of absolute retracts such that each bonding map is a retraction. We show that Y is the almost continuous retract of the Hilbert cube. It follows that Y, the cone over Y, the suspension of Y, and the product of Y with any absolute retract must have the fixed point property.

Introduction. Throughout this paper X and Y will denote topological spaces. A *map* is a continuous function. When $f: X \to Y$ may not be continuous, we refer to it simply as the function f. A space X has the *fixed point property* if for each map $f: X \to X$ there exist $x \in X$ such that $f(x) = x$. A *continuum* is a compact connected metric topological space. An *absolute retract* (AR) is a retract of the Hilbert cube.

The fixed point property has been the subject of intense investigation. Many surprising results have been revealed, but many questions still remain unanswered.

J. Stallings [**7**] defined a class of functions, which he named almost continuous, for the purpose of studying the fixed point property.

The *graph* of a function $f: X \to Y$ is the subset of $X \times Y$ consisting of the points $(x, f(x))$; this set will be symbolized $\Gamma(f)$.

DEFINITION 1 [**7**, p. 252]. A function $f: X \to Y$ is *almost continuous* if for each open subset U of $X \times Y$ such that $\Gamma(f) \subset U$, there exists a map $g: X \to Y$ such that $\Gamma(g) \subset U$.

DEFINITION 2. If $Y \subset X$ and $r: X \to Y$ is an almost continuous function such that $r(X) = Y$ and for all $x \in Y$, $r(x) = x$, then r is called a *quasi retraction* and Y is called a *quasi retract* of X.

DEFINITION 3. If $Y \subset X$ and $r: X \to Y$ is an almost continuous function such that $r(X) = Y$ and $r(x) = x$ for all $x \in Y$, then r is called an *almost continuous retraction* and Y is called an *almost continuous retract* of X.[1]

DEFINITION 4 [**1**, P. 48]. A compact metric space Y is an *absolute quasi retract* (AQR) if Y is homeomorphic to a quasi retract of the Hilbert cube.

THEOREM 1 [**1**]. *Every AQR has the fixed point property.*

THEOREM 2 [**1**]. *If X is an AR and Y an AQR, then $X \times Y$ is an AQR.*

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[1] In the literature both almost continuous retractions and quasi retractions have been called almost continuous retractions. We make the distinction because even though almost continuous retractions are quasi retractions, the converse is not true. We refer the interested reader to [**3**] and [**1**].

COROLLARY. *If X is an AR and Y an AQR, then $X \times Y$ has the fixed point property.*

Let $I = \{t: 0 \leq t \leq 1\}$. For any space X the *cone* TX over X is the quotient space $(X \times I)/R$, where R is the equivalence relation $(x,t) \sim (y,s)$ if and only if $t = s = 1$, or $x = y$ and $t = s$, for all $x, y \in X$ and $s, t \in I$. The *suspension* SX of X is the quotient space $(X \times I)/Q$, where Q is the equivalence relation $(x,t) \sim (y,s)$ if and only if $t = s = 1$ or $t = s = 0$ or $x = y$ and $t = s$, for $x, y \in X$ and $s, t \in I$.

THEOREM 3 [**1**]. *If X is an AQR, then TX and SX are AQRs.*

COROLLARY. *If X is an AQR, then TX and SX have the fixed point property.*

Given a sequence of topological spaces $Y_1, Y_2, \ldots$ and maps $g[j,i]: Y_j \to Y_i$ where $i < j$, the set $\{Y_i; g[j,k]\}$ is called an *inverse system.* The maps $g[j,k]$ are called the *bonding maps* of the inverse system. Let Y be the set of all points $\langle y_1, y_2, \ldots \rangle$ in $\prod Y_i$ such that, for $i < j$, $y_i = g[j,i](y_j)$. The set Y, considered as a subspace of $\prod Y_i$, is called the *inverse limit* of $\{Y_i; g[j,k]\}$.

We prove that if Y is the inverse limit of $\{Y_i; r[j,i]\}$ where each Y_i is an absolute retract and each bonding map $r[j,i]$ is a retraction, then Y is homeomorphic to an almost continuous retract of the Hilbert cube. Therefore Y is an AQR.

The main results. An almost continuous function $f: X \to Y$ is a function approximated by maps in the sense of Definition 1. If however, X and Y are compact metric spaces there is a characterization of almost continuity in terms of sequences of maps.

DEFINITION 5. A sequence $\{f_n\}$ of function of X into Y *almost continuously approximates* a function $f: X \to Y$ if for every sequence $\{x_n\} \subset X$, such that $f_n(x_n) \neq f(x_n)$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ and $x \in X$ such that $x_{n_i} \to x$ and $f_{n_i}(x_{n_i}) \to f(x)$.

THEOREM 4 [**1**]. *Assume X and Y are compact metric spaces. Then $f: X \to Y$ is almost continuous if and only if there exists a sequence of maps $f_n: X \to Y$ such that $\{f_n\}$ almost continuously approximates f.*

Henceforth we will use Theorem 4 for proving that a given function is almost continuous.

THEOREM 5. *Let (X, ρ) be a compact metric space. Suppose $X = X_1 \supset X_2 \supset \cdots \supset X_n \supset X_{n+1} \supset \cdots$ is a decreasing sequence of compact subsets of X such that $X_{n+1} \subset \operatorname{int} X_n$. Also suppose that $M_1 \subset M_2 \subset \cdots \subset M_n \subset M_{n+1} \subset \cdots$ is an increasing sequence of absolute retracts such that $M_n \subset X_n$. If there exist maps $r_n: X_n \to M_n$ such that $\rho(x, r_n(x)) < 1/n$, then $\bigcap X_i$ is an almost continuous retract of X.*

PROOF. Let $f_1 = r_1$. Thus f_1 is a map of X_1 into M_1. Inductively we define maps $f_n: X_1 \to M_n$ as follows. Given $f_k: X_1 \to M_k$ let G_k be open in X_1 such that $X_{k+1} \subset G_k \subset X_k$ and let $F_k = (X_1 \sim G_k) \cup X_{k+1}$. Next define a map $g_k: F_k \to M_{k+1}$ as follows:

$$g_k(x) = \begin{cases} f_k(x) & \text{if } x \in X_1 \sim G_k, \\ r_{k+1}(x) & \text{if } x \in X_{k+1}. \end{cases}$$

Now extend g_k to a map $f_{k+1}: X_1 \to M_{k+1}$.

Consider the function $r: X_1 \to \bigcap X_i$ defined as

$$r(x) = \begin{cases} x & \text{if } x \in \bigcap X_i, \\ f_{n+1}(x) & \text{if } x \in X_n \sim X_{n+1}. \end{cases}$$

We prove that $r: X_1 \to \bigcap X_i$ is almost continuous by showing that $\{f_n\}$ almost continuously approximates r.

Choose a sequence $\{x_n\}$ in X_1, such that $f_n(x_n) \neq r(x_n)$. Since $r(x) = f_n(x)$ for every $x \in X \sim X_n$, we conclude that $x_n \in X_n$. Therefore $f_n(x_n) = r_n(x_n)$.

Let $\{x_{n_i}\}$ be a convergent subsequence of $\{x_n\}$, with $x_{n_i} \to x \in \bigcap X_i$. Since $\rho(r_n(x_n), x_n) < 1/n$, we conclude that $r_{n_i}(x_{n_i}) \to x$, hence $f_{n_i}(x_{n_i}) \to x$. Also $r(x) = x$ because $x \in \bigcap X_i$, it follows that $f_{n_i}(x_{n_i}) \to r(x)$. Therefore $\{f_n\}$ almost continuously approximates r. Thus $r: X \to \bigcap X_i$ is an almost continuous retraction. And the proof of Theorem 5 is complete.

Next we consider inverse systems of absolute retracts with bonding maps which are retractions.

THEOREM 6. *Let $\{Y_i; r[j,k]\}$ be an inverse system such that each Y_i is an absolute retract, and if $i < j$ then $Y \subset Y_j$, and $r[j,k]: Y_j \to Y_i$ is a retraction. Then the inverse limit of $\{Y_i; r[j,i]\}$ is an almost continuous retract of $\prod Y_i$.*

PROOF. Let Y denote the inverse limit of $\{Y_i; r[j,i]\}$. Let $X_1 = \prod Y_i$. For $n = 2, 3, 4, \ldots$ let X_n be the set of all points $\langle x_i \rangle = \langle x_1, x_2, \ldots \rangle$ in $\prod Y_i$ such that if $i < j \leq n$ then $\rho_i(x_i, r[j,i](x_j)) \leq 2^{-n}$, where ρ_i denotes the metric of Y_i.

By a routine argument, each X_n is compact, and each $X_{n+1} \subset \operatorname{int} X_n$; and, clearly, $Y = \bigcap X_n$.

For $n = 1, 2, 3, \ldots$ let M_n be the set of all points $\langle x_i \rangle$ in $\prod Y_i$ such that if $i < n$ then $x_i = r[n,i](x_n)$ and if $i \geq n$ then $x_i = x_n$. Then $M_n \subset M_{n+1}$, and each M_n is an AR since it is homeomorphic to Y_n.

Next we define the maps $r_n: X_n \to M_n$ as follows:

$$r_n\langle x_i \rangle = \begin{cases} r[n,i](x_n) & \text{if } i < n, \\ x_n & \text{if } i \geq n. \end{cases}$$

Let ρ denote the metric of $\prod Y_i$, defined by $\rho(\langle x_i \rangle, \langle y_i \rangle) = \sum 2^{-i} \rho_i(x_i, y_i)$. Then for $\langle x_i \rangle \in X_n$,

$$\rho(\langle x_i \rangle, r_n \langle x_i \rangle) = \sum_{i=1}^{n-1} \frac{\rho_i(x_i, r[n,i](x_n))}{2^i} + \sum_{i=n}^{\infty} \frac{\rho_i(x_i, x_n)}{2^i}.$$

We may assume with no loss of generality that $\rho_i(x,y) \leq \frac{1}{2}$ for each $x, y \in Y_i$ and $i = 1, 2, 3, \ldots$. Hence

$$\rho(\langle x_i \rangle, r_n \langle x_i \rangle) \leq \frac{1}{2^n} \sum_{i=1}^{n-1} \frac{1}{2^i} + \frac{1}{2^n} < \frac{1}{2^{n-1}} < \frac{1}{n}.$$

It follows from Theorem 5 that Y is an almost continuous retract of $\prod Y_i$.

COROLLARY. *Let Y be the inverse limit of $\{Y_i; r[j,i]\}$ as described in Theorem 6. Then Y is homeomorphic to an almost continuous retract of the Hilbert cube H.*

PROOF. Since each Y_i is an AR, then $\prod Y_i$ is an AR [**2**]. Let $r: H \to \prod Y_i$ be a retraction and let $q: \prod Y_i \to Y$ be an almost continuous retraction. Then by Proposition 4 of [**7**, p. 261], $qr: H \to Y$ is an almost continuous retraction.

COROLLARY. *Let Y be the inverse limit of $\{Y_i; r[j,i]\}$ as described in Theorem 6. Then Y, the cone over Y, the suspension of Y, and the product of Y with an AR must have the fixed point property.*

PROOF. This follows from the preceding corollary and the corollaries of Theorems 2 and 3.

Concluding remarks and questions. A map $f: X \to Y$ is *universal* if for any continuous map $: X \to Y$ there exists $x \in X$ such that $g(x) = f(x)$. W. Holsztynski [**4**] has shown that if Y is the inverse limit of $\{Y_i; g[j,i]\}$, where each Y_i is an AR and each bonding map $g[j,i]$ is universal, then Y has the fixed point property. It is easy to see that if Y has the fixed point property, and if $r: X \to Y$ is a retraction, then r is universal. Thus our last corollary also follows from Holsztynski's theorem.

In view of Holsztynski's result one might hope to get a stronger version of Theorem 6, by allowing the bonding maps $r[j,i]$ to be universal. However, this cannot be done. K. Kellum [**5**] proved that: Given a 2nd countable space Y, there exists a Peano continuum P such that Y is the image of a surjective almost continuous function $f: P \to Y$ if and only if Y is almost Peano. That Y is *almost Peano* means that for each finite collection of nonempty open subsets of Y, there is a Peano continuum in Y which intersects each of them. It follows that if Y is the almost continuous image of the Hilbert cube then Y is almost Peano. However, the pseudoarc [**6**] is not almost Peano, since it contains no Peano subcontinuum. But the pseudoarc is the inverse limit of arcs. And since every map of an arc onto itself is universal [**4**], the pseudoarc is the inverse limit of absolute retracts with universal bonding maps. We conclude that there exists an example of a space, which is the inverse limit of absolute retracts with universal bonding maps, that is not an almost continuous retract of the Hilbert cube.

However, it is not known if the pseudoarc is an AQR. Consequently, it is natural to ask the following

Question. If Y is the inverse limit of $\{Y_i; g[j,i]\}$, where each Y_i is an absolute retract and each $g[j,i]$ is universal, is Y an AQR?

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REFERENCES

1. V. N. Akis, *Fixed point theorems and almost continuity*, Fund. Math. **121** (1984), 134–142.
2. K. Borsuk, *Theory of retracts*, Monograf. Mat., Tom 44, PWN, Warsaw, 1967.
3. B. D. Garrett, *Almost continuous retracts*, General Topology and Modern Analysis, Academic Press, New York, 1981, pp. 229–238.
4. W. Holsztynski, *Universal mappings and fixed point theorems*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **15** (1967), 433–438.
5. K. R. Kellum, *Almost continuous images of Peano continua*, Topology Appl. **11** (1980), 293–296.
6. E. E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua*, Trans. Amer. Math. Soc. **63** (1948), 581–594.
7. J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. **47** (1959), 249–263.

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