INVERSE SYSTEMS OF ABSOLUTE RETRACTS AND ALMOST CONTINUITY

VLADIMIR N. AKIS

ABSTRACT. Suppose that $Y$ is the inverse limit of a sequence of absolute retracts such that each bonding map is a retraction. We show that $Y$ is the almost continuous retract of the Hilbert cube. It follows that $Y$, the cone over $Y$, the suspension of $Y$, and the product of $Y$ with any absolute retract must have the fixed point property.

Introduction. Throughout this paper $X$ and $Y$ will denote topological spaces. A map is a continuous function. When $f: X \to Y$ may not be continuous, we refer to it simply as the function $f$. A space $X$ has the fixed point property if for each map $f: X \to X$ there exist $x \in X$ such that $f(x) = x$. A continuum is a compact connected metric topological space. An absolute retract (AR) is a retract of the Hilbert cube.

The fixed point property has been the subject of intense investigation. Many surprising results have been revealed, but many questions still remain unanswered. J. Stallings [7] defined a class of functions, which he named almost continuous, for the purpose of studying the fixed point property.

The graph of a function $f: X \to Y$ is the subset of $X \times Y$ consisting of the points $(x, f(x))$; this set will be symbolized $\Gamma(f)$.

DEFINITION 1 [7, p. 252]. A function $f: X \to Y$ is almost continuous if for each open subset $U$ of $X \times Y$ such that $\Gamma(f) \subseteq U$, there exists a map $g: X \to Y$ such that $\Gamma(g) \subseteq U$.

DEFINITION 2. If $Y \subseteq X$ and $r: X \to Y$ is an almost continuous function such that $r(X) = Y$ and for all $x \in Y$, $r(x) = x$, then $r$ is called a quasi retraction and $Y$ is called a quasi retract of $X$.

DEFINITION 3. If $Y \subseteq X$ and $r: X \to Y$ is an almost continuous function such that $r(X) = Y$ and $r(x) = x$ for all $x \in Y$, then $r$ is called an almost continuous retraction and $Y$ is called an almost continuous retract of $X$.\footnote{In the literature both almost continuous retractions and quasi retractions have been called almost continuous retractions. We make the distinction because even though almost continuous retractions are quasi retractions, the converse is not true. We refer the interested reader to [3] and [1].}

DEFINITION 4 [1, P. 48]. A compact metric space $Y$ is an absolute quasi retract (AQR) if $Y$ is homeomorphic to a quasi retract of the Hilbert cube.

THEOREM 1 [1]. Every AQR has the fixed point property.

THEOREM 2 [1]. If $X$ is an AR and $Y$ an AQR, then $X \times Y$ is an AQR.

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COROLLARY. If $X$ is an AR and $Y$ an AQR, then $X \times Y$ has the fixed point property.

Let $I = \{t: 0 \leq t \leq 1\}$. For any space $X$ the cone $TX$ over $X$ is the quotient space $(X \times I) / R$, where $R$ is the equivalence relation $(x,t) \sim (y,s)$ if and only if $t = s = 1$, or $x = y$ and $t = s$, for all $x, y \in X$ and $s, t \in I$. The suspension $SX$ of $X$ is the quotient space $(X \times I) / Q$, where $Q$ is the equivalence relation $(x,t) \sim (y,s)$ if and only if $t = s = 1$ or $t = s = 0$ or $x = y$ and $t = s$, for $x, y \in X$ and $s, t \in I$.

**Theorem 3 [1].** If $X$ is an AQR, then $TX$ and $SX$ are AQRs.

**Corollary.** If $X$ is an AQR, then $TX$ and $SX$ have the fixed point property.

Given a sequence of topological spaces $Y_1, Y_2, \ldots$ and maps $g[j,i]: Y_j \rightarrow Y_i$ where $i < j$, the set $\{Y_i; g[j,i]\}$ is called an inverse system. The maps $g[j,k]$ are called the bonding maps of the inverse system. Let $Y$ be the set of all points $(y_1, y_2, \ldots)$ in $\prod Y_i$ such that, for $i < j$, $y_i = g[j,i](y_j)$. The set $Y$, considered as a subspace of $\prod Y_i$, is called the inverse limit of $\{Y_i; g[j,k]\}$.

We prove that if $Y$ is the inverse limit of $\{Y_i; r[j,i]\}$ where each $Y_i$ is an absolute retract and each bonding map $r[j,i]$ is a retraction, then $Y$ is homeomorphic to an almost continuous retract of the Hilbert cube. Therefore $Y$ is an AQR.

**The main results.** An almost continuous function $f: X \rightarrow Y$ is a function approximated by maps in the sense of Definition 1. If however, $X$ and $Y$ are compact metric spaces there is a characterization of almost continuity in terms of sequences of maps.

**Definition 5.** A sequence $\{f_n\}$ of function of $X$ into $Y$ almost continuously approximates a function $f: X \rightarrow Y$ if for every sequence $\{x_n\} \subset X$, such that $f_n(x_n) \neq f(x_n)$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ and $x \in X$ such that $x_{n_i} \rightarrow x$ and $f_{n_i}(x_{n_i}) \rightarrow f(x)$.

**Theorem 4 [1].** Assume $X$ and $Y$ are compact metric spaces. Then $f: X \rightarrow Y$ is almost continuous if and only if there exists a sequence of maps $f_n: X \rightarrow Y$ such that $\{f_n\}$ almost continuously approximates $f$.

Henceforth we will use Theorem 4 for proving that a given function is almost continuous.

**Theorem 5.** Let $(X, p)$ be a compact metric space. Suppose $X = X_1 \supset X_2 \supset \cdots \supset X_n \supset X_{n+1} \supset \cdots$ is a decreasing sequence of compact subsets of $X$ such that $X_{n+1} \subset \text{int}X_n$. Also suppose that $M_1 \subset M_2 \subset \cdots \subset M_n \subset M_{n+1} \subset \cdots$ is an increasing sequence of absolute retracts such that $M_n \subset X_n$. If there exist maps $r_n: X_n \rightarrow M_n$ such that $p(x, r_n(x)) < 1/n$, then $\bigcap X_i$ is an almost continuous retract of $X$.

**Proof.** Let $f_1 = r_1$. Thus $f_1$ is a map of $X_1$ into $M_1$. Inductively we define maps $f_n: X_1 \rightarrow M_n$ as follows. Given $f_k: X_1 \rightarrow M_k$ let $G_k$ be open in $X_1$ such that $X_{k+1} \subset G_k \subset X_k$ and let $F_k = (X_1 \sim G_k) \cup X_{k+1}$. Next define a map $g_k: F_k \rightarrow M_{k+1}$ as follows:

$$g_k(x) = \begin{cases} f_k(x) & \text{if } x \in X_1 \sim G_k, \\ r_{k+1}(x) & \text{if } x \in X_{k+1}. \end{cases}$$

Now extend $g_k$ to a map $f_{k+1}: X_1 \rightarrow M_{k+1}$.
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Consider the function \( r: X_i \rightarrow \bigcap X_i \) defined as

\[
  r(x) = \begin{cases} 
    x & \text{if } x \in \bigcap X_i, \\
    f_{n+1}(x) & \text{if } x \in X_n \sim X_{n+1}.
  \end{cases}
\]

We prove that \( r: X_1 \rightarrow \bigcap X_i \) is almost continuous by showing that \( \{f_n\} \) almost continuously approximates \( r \).

Choose a sequence \( \{x_n\} \) in \( X_1 \), such that \( f_n(x_n) \rightarrow r(x_n) \). Since \( r(x) = f_n(x) \) for every \( x \in X \sim X_n \), we conclude that \( x_n \in X_n \). Therefore \( f_n(x_n) = r_n(x_n) \).

Let \( \{x_{n_i}\} \) be a convergent subsequence of \( \{x_n\} \), with \( x_{n_i} \rightarrow x \in \bigcap X_i \). Since \( \rho(r_n(x_{n_i}), x_{n_i}) < 1/n \), we conclude that \( r_n(x_{n_i}) \rightarrow x \), hence \( f_n(x_{n_i}) \rightarrow x \). Also \( r(x) = x \) because \( x \in \bigcap X_i \), it follows that \( f_n(x_{n_i}) \rightarrow r(x) \). Therefore \( \{f_n\} \) almost continuously approximates \( r \). Thus \( r: X \rightarrow \bigcap X_i \) is an almost continuous retraction. And the proof of Theorem 5 is complete.

Next we consider inverse systems of absolute retracts with bonding maps which are retractions.

**Theorem 6.** Let \( \{Y_i; r[j, k]\} \) be an inverse system such that each \( Y_i \) is an absolute retract, and if \( i < j \) then \( Y_i \subseteq Y_j \), and \( r[j, k]: Y_j \rightarrow Y_i \) is a retraction. Then the inverse limit of \( \{Y_i; r[j, i]\} \) is an almost continuous retract of \( \prod Y_i \).

**Proof.** Let \( Y \) denote the inverse limit of \( \{Y_i; r[j, i]\} \). Let \( X_1 = \prod Y_i \). For \( n = 2, 3, 4, \ldots \) let \( X_n \) be the set of all points \( \langle x_i \rangle = \langle x_1, x_2, \ldots \rangle \) in \( \prod Y_i \) such that if \( i < j \leq n \) then \( \rho_i(x_i, r[j, i](x_j)) \leq 2^{-n} \), where \( \rho_i \) denotes the metric of \( Y_i \).

By a routine argument, each \( X_n \) is compact, and each \( X_{n+1} \subseteq \text{int} X_n \); and, clearly, \( Y = \bigcap X_n \).

For \( n = 1, 2, 3, \ldots \) let \( M_n \) be the set of all points \( \langle x_i \rangle \) in \( \prod Y_i \) such that if \( i < n \) then \( x_i = r[n, i](x_n) \) and if \( i \geq n \) then \( x_i = x_n \). Then \( M_n \subseteq M_{n+1} \), and each \( M_n \) is an AR since it is homeomorphic to \( Y_n \).

Next we define the maps \( r_n: X_n \rightarrow M_n \) as follows:

\[
  r_n(x_i) = \begin{cases} 
    r[n, i](x_n) & \text{if } i < n, \\
    x_n & \text{if } i \geq n.
  \end{cases}
\]

Let \( \rho \) denote the metric of \( \prod Y_i \), defined by \( \rho(\langle x_i \rangle, \langle y_i \rangle) = \sum 2^{-i} \rho_i(x_i, y_i) \). Then for \( \langle x_i \rangle \in X_n \),

\[
  \rho(\langle x_i \rangle, r_n(x_i)) = \sum_{i=1}^{n-1} \frac{\rho_i(x_i, r[n, i](x_n))}{2^i} + \sum_{i=n}^{\infty} \frac{\rho_i(x_i, x_n)}{2^i}.
\]

We may assume with no loss of generality that \( \rho_i(x, y) \leq \frac{1}{2} \) for each \( x, y \in Y_i \) and \( i = 1, 2, 3, \ldots \). Hence

\[
  \rho(\langle x_i \rangle, r_n(x_i)) \leq \frac{1}{2^n} \sum_{i=1}^{n-1} \frac{1}{2^i} + \frac{1}{2^n} < \frac{1}{2^{n-1}} < \frac{1}{n}.
\]

It follows from Theorem 5 that \( Y \) is an almost continuous retract of \( \prod Y_i \).

**Corollary.** Let \( Y \) be the inverse limit of \( \{Y_i; r[j, i]\} \) as described in Theorem 6. Then \( Y \) is homeomorphic to an almost continuous retract of the Hilbert cube \( H \).

**Proof.** Since each \( Y_i \) is an AR, then \( \prod Y_i \) is an AR [2]. Let \( r: H \rightarrow \prod Y_i \) be a retraction and let \( q: \prod Y_i \rightarrow Y \) be an almost continuous retraction. Then by Proposition 4 of [7, p. 261], \( qr: H \rightarrow Y \) is an almost continuous retraction.
COROLLARY. Let $Y$ be the inverse limit of $\{Y_i; r[j, i]\}$ as described in Theorem 6. Then $Y$, the cone over $Y$, the suspension of $Y$, and the product of $Y$ with an AR must have the fixed point property.

PROOF. This follows from the preceding corollary and the corollaries of Theorems 2 and 3.

Concluding remarks and questions. A map $f: X \to Y$ is universal if for any continuous map $g: X \to Y$ there exists $x \in X$ such that $g(x) = f(x)$. W. Holsztynski [4] has shown that if $Y$ is the inverse limit of $\{Y_i; g[j, i]\}$, where each $Y_i$ is an AR and each bonding map $g[j, i]$ is universal, then $Y$ has the fixed point property. It is easy to see that if $Y$ has the fixed point property, and if $r: X \to Y$ is a retraction, then $r$ is universal. Thus our last corollary also follows from Holsztynski's theorem.

In view of Holsztynski's result one might hope to get a stronger version of Theorem 6, by allowing the bonding maps $r[j, i]$ to be universal. However, this cannot be done. K. Kellum [5] proved that: Given a 2nd countable space $Y$, there exists a Peano continuum $P$ such that $Y$ is the image of a surjective almost continuous function $f: P \to Y$ if and only if $Y$ is almost Peano. That $Y$ is almost Peano means that for each finite collection of nonempty open subsets of $Y$, there is a Peano continuum in $Y$ which intersects each of them. It follows that if $Y$ is the almost continuous image of the Hilbert cube then $Y$ is almost Peano. However, the pseudoarc [6] is not almost Peano, since it contains no Peano subcontinuum. But the pseudoarc is the inverse limit of arcs. And since every map of an arc onto itself is universal [4], the pseudoarc is the inverse limit of absolute retracts with universal bonding maps. We conclude that there exists an example of a space, which is the inverse limit of absolute retracts with universal bonding maps, that is not an almost continuous retract of the Hilbert cube.

However, it is not known if the pseudoarc is an AQR. Consequently, it is natural to ask the following

Question. If $Y$ is the inverse limit of $\{Y_i; g[j, i]\}$, where each $Y_i$ is an absolute retract and each $g[j, i]$ is universal, is $Y$ an AQR?

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, CALIFORNIA STATE UNIVERSITY, LOS ANGELES, CALIFORNIA 90032