CORRIGENDUM TO “EMBEDDINGS IN $G(1, 3)$”

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As was pointed out by N. Goldstein in MR 84i, Lemma 2.2 in [3] is incorrect. We give here a revised version of the main theorem in [3]. In the proof we avoid Lemma 2.2 and use instead [2, Theorem 2.3 and Corollary 2.4], [1, Corollary IV.18], and Castelnuovo’s bound on the genus of a curve in $\mathbb{P}^n$. In the revised theorem the restriction to surfaces that are not projections from a higher $\mathbb{P}^n$ is removed.

**Theorem.** Let $Y$ be a nonsingular surface in $G \subset \mathbb{P}^5$ of degree $d \leq 8$, and let $(a, b)$ be its class in the Chow ring $A(G)$ of $G$. Then one of the following holds:

(i) $d = 1$, $(a, b) = (1, 0)$, $Y = \mathbb{P}^2$;
(ii) $d = 2$, $(a, b) = (1, 1)$, $Y = F_0$;
(iii) $d = 3$, $(a, b) = (2, 1)$, $Y = F_1$;
(iv) $d = 4$, and either $(a, b) = (2, 2)$ and $Y = F_0$, $F_2$, or the del Pezzo $S_4$, or else $(a, b) = (1, 3)$ and $Y = \text{the Veronese surface}$;
(v) $d = 5$, $(a, b) = (2, 3)$ and $Y = F_6$ with 3 or 7 points blown up;
(vi) $d = 6$, $(a, b) = (3, 3)$, and either $Y = F_6$ with 2 or 6 points blown up, or $Y$ is a geometrically ruled surface with $p_a = -1$, or $Y = G \cap \mathbb{P}^4 \cap S_3$ and is a K3 surface;
(vii) $d = 7$, and either $Y$ is geometrically ruled with $p_a = -3$, or $Y$ is ruled with 2 points blown up with $p_a = -3$, or $Y$ is ruled with 4 or 6 points blown up with $p_a = -1$, or $Y = F_6$ with 8 points blown up, or $Y$ is the cubic surface with 5 points blown up, or $K^2 = -12 + 6p_a$;
(viii) $d = 8$, and either $(a, b) = (4, 4)$, and $Y = F_6$ with 6 or 10 points blown up, or $Y$ is geometrically ruled with $p_a = -3$, or $Y$ is ruled with 4 points blown up with $p_a = -1$, or $Y$ is a complete intersection of three quadrics, or $Y = G \cap \mathbb{P}^4 \cap S_4$ is a surface of general type; or $(a, b) = (2, 6)$ and $Y$ is geometrically ruled with $p_a = -1$; or $(a, b) = (3, 5)$ and $Y$ is ruled with 3 points blown up with $p_a = -1$.

To facilitate the reading of the proof, we list here some facts we use.

I. **CASTELNUOVO’S BOUND ON THE GENUS.** Let $C$ be a nonsingular curve of degree $d$, $C \subset \mathbb{P}^n$, not lying in any $\mathbb{P}^{n-1}$. Then $g \leq m(m - 1)(n - 1)/2 + me$ where $m = \lfloor d - 1/n - 1 \rfloor$ and $e = (d - 1) - m(n - 1)$.

II. **CRITERION FOR A SURFACE TO BE RULED** [1, Corollary VI.18]. $S$ is ruled if and only if there is a curve $C \subset S$, not an exceptional divisor, such that $C \cdot K < 0$. 

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533
III. CURVES ON NONRATIONAL RULED SURFACES. (1) [2, Theorem 2.3]: Let \( \pi: Y \to T \) be a nonrational ruled surface, \( C \subset Y \) an irreducible curve, and \( m \) the degree of \( \pi: C \to T \), with \( m > 1 \). Then

\[
C^2 \leq \frac{2m}{m-1}(g(C) - 1).
\]

(2) [2, Corollary 2.4]: Let \( C \) and \( Y \) be as above. Then either

(a) \( C \equiv T \) and the embedding of \( C \subset Y \) is equivalent to a section of the geometrically ruled surface \( \pi: \mathbb{P}(E) \to T \) and \( (C^2)_Y = (T^2)(\pi(E)) \), or

(b) \( C^2 \leq 4g(C) - 4 \).

IV. FURTHER EQUATIONS. With the notation as in [3, p. 584], let \( s \) be the number of points in \( X \) with \( r_p = 1 \), \( t \) the number of points in \( X \) with \( r_p = 2 \) and \( \varphi^{-1}(p) = E \), one exceptional divisor, and \( u \) the number of points in \( X \) with \( r_p = 2 \) and \( \varphi^{-1}(p) = E_1 \cup E_2 \), two exceptional divisors. Assume that for all \( p \in X \), \( r_p \leq 2 \).

Then

(3) \( X^2 = m(2n - m) = C^2 + s + At + 5u \),

(4) \( 2g(X) - 2 = X^2 - X^2/m - 2m(p_a(Y_0) + 1) \),

(5) \( g(X) = g(C) + t + u \),

(6) \( K^2_{Y_0} = K^2_Y + r \),

(7) \( r = s + t + 2u \).

PROOF OF THEOREM. (i) and (ii) are obvious.

(iii) If \( d = 3 \), then \( Y \subset \mathbb{P}^4 \); by I, \( g(C) = 0 \), \( HK = -5 \); II implies that \( Y \) is ruled, and III implies \( p_a = 0 \), and the proof in [3] for \( d = 3 \) applies.

(iv) If \( d = 4 \) and \( Y \subset \mathbb{P}^4 \), then, by I, \( g(C) = 0 \), \( HK = -6 \); II implies that \( Y \) is ruled, and III implies that \( p_a = 0 \), and the proof in [3] for \( d = 4 \) applies.

(v) If \( d = 5 \) and \( Y \subset \mathbb{P}^4 \), then, by I, \( g(C) \leq 1 \). Suppose \( g(C) = 0 \). Then \( KH = -7 \) and \( Y \) is a rational surface by II and III, but [3, (2)] leads to a contradiction. Therefore \( g(C) = 1 \), \( KH = -5 \), and II implies that \( Y \) is ruled. If \( p_a \neq 0 \), then by III(2), \( p_a = -1 \), \( C \) is a section, \( X^2 = C^2 \), and \( Y \) is geometrically ruled. Hence, \( K^2 = 0 \), and this contradicts [3, (2)]. Therefore \( p_a = 0 \), and the proof in [3] for \( d = 5 \) \( Y \subset \mathbb{P}^4 \) applies.

If \( d = 5 \) and \( Y \subset \mathbb{P}^4 \), \( g(C) = 0 \) or 2 by [3, Proposition 1.2]. Suppose \( g(C) = 0 \). Then \( KH = -7 \) and \( Y \) is a rational ruled surface by II and III. Finally, by [3, (3) and (2)] we get a contradiction. Hence, \( g(C) = 2 \), \( KH = -3 \), and \( Y \) is ruled. If \( p_a \neq 0 \), then, by III, \( p_a = -2 \), \( K^2 = -8 \), and this contradicts [3, (2)]. Hence, \( g(C) = 2 \), \( p_a = 0 \), and the proof in [3] for \( d = 5 \) \( Y \subset \mathbb{P}^4 \) applies.

(vi) If \( d = 6 \) and \( Y \subset \mathbb{P}^4 \), then, by I, \( g(C) \leq 2 \). If \( g(C) = 0 \), then \( KH = -8 \), and II and III imply that \( Y \) is a rational ruled surface, and this contradicts [3, (2)]. If \( g(C) = 1 \), then \( KH = -6 \), and \( Y \) is ruled. If \( p_a = 0 \), then by [3, Proposition 1.3] \( m \geq 3 \) and \( r_p = 1 \), and by [3, (2)] \( K^2 = 7 \) or 6. If \( K^2 = 7 \), then, by (3) and (6), \( X^2 = 7 \); by (5), \( g(X) = g(C) = 1 \); and by (4), \( 0 = 7 - 7/m - 2m \), which is
impossible. Hence, \( K^2 = 6 \), \((a, b) = (3,3)\), and, by (4), \(0 = 8 - 8/m - 2m\), which implies \( m = 2 \). If \( p_a \neq 0 \), then, by II and III, \( Y \) is geometrically ruled with \( p_a = -1 \), \( m = 1 \), \( e = 0 \), \( n = 3 \), and \((a, b) = (3,3)\). If \( g(C) = 2 \) with \( p_a \neq 0 \), then, by II and III, \( Y \) is geometrically ruled with \( p_a = -2 \), and this contradicts [3, (2)]. Hence, if \( g(C) = 2 \), then \( Y \) is a rational ruled surface and the proof in [3] for \( d = 6 \) applies.

(vii) If \( d = 7 \) and \( Y \subset \mathbb{P}^4 \), then, by I, \( g(C) \leq 3 \). If \( g(C) = 0 \), then \( KH = -9 \) and \( Y \) is a rational ruled surface, which contradicts [3, (2)]. If \( g(C) = 1 \), then \( KH = -7 \), and if \( p_a = 0 \), then by [3, (2)], \((a, b) = (3,4)\) and \( Y \) is geometrically ruled, and, by (4), \( 2g(C) - 2 = 0 = 7 - 7/m - 2m \), which is impossible. Hence, if \( g(C) = 1 \), then \( p_a \neq 0 \), and, by III, \( p_a = -1 \), \( K^2 = 0 \), and this contradicts [3, (2)].

If \( g(C) = 2 \) and \( p_a = 0 \), then by [3, (2)], \( K^2 = 6 \) or 4, and, by [3, Proposition 1.3], \( m \leq 3 \), whence \( r = 1 \). Hence, by (3) \( X^2 = 9 \) or 11 and, by (4), \( 2 = 9 - 9/m - 2m \) or \( 2 = 11 - 11/m - 2m \), and both are impossible. Hence, if \( g(C) = 2 \), then \( p_a \neq 0 \), and, by II and III, \( Y \) is geometrically ruled with \( p_a = -2 \), \( C^2 = 7 = 2n - e \), and \( T_0 \cdot C = n - e \geq 5 \), since \( T_0 \) is a nonsingular curve of genus 2; but this contradicts the fact that \( e \geq p_a = -2 \).

Hence, if \( d = 7 \), then \( g(C) = 3 \) and \( KH = -3 \). If \( p_a \neq 0 \), then, by III, \( m \leq 2 \). If \( m = 1 \), then \( r_p = 1 \), \( g(X) = g(C) \), and, by (4), \( p_a = -3 \). Hence, by [3, (2)], \( Y \) is of type (2, 5) and is geometrically ruled with \( e = -1 \), \( n = 3 \), or of type (3, 4) with \( K^2 = -18 \). If \( m = 2 \), by (3)-(5), we get \( 9 + 8p_a = s + u \geq 0 \). Therefore \( p_a = -1 \) and, by [3, (2)], \( K^2 = -4 \) or \(-6 \). If \( p_a = 0 \), then the proof in [3] for \( d = 7 \) applies.

(viii) If \( d = 8 \) and \( Y \subset \mathbb{P}^4 \), then, by I, \( g(C) \leq 5 \). Suppose \( g(C) = 0 \). Then \( p_a = 0 \) and [3, (2)] leads us to a contradiction. If \( g(C) = 1 \), then by [3, (2)], \( p_a < 0 \), and, by III, \( p_a = -1 \); hence \( K^2 \leq 0 \), and this contradicts [3, (2)]. If \( g(C) = 2 \) with \( p_a = 0 \), then, by [3, (2)], \( K^2 = 6 \) or 7, and by [3, Proposition 1.3], \( m < 4 \); hence, \( r_p = 1 \), and, by (4), \( 2 = 10 - 10/m - 2m \) or \( 2 = 9 - 9/m - 2m \), and both are impossible. Hence, if \( g(C) = 2 \), then \( p_a < 0 \), and, by III, \( C \) is a section, \( Y \) is geometrically ruled with \( p_a = -2 \), and \( K^2 = -8 \); but this is impossible by [3, (2)]. Therefore if \( d = 8 \) and \( Y \subset \mathbb{P}^4 \), then \( 3 \leq g(C) \leq 5 \).

If \( g(C) = 3 \) with \( p_a = 0 \), then, by [3, Proposition 1.3], \( m \leq 4 \). If \( m < 4 \), then \( r_p = 1 \), and by (5), \( g(X) = 3 \), and, by (4), \( 4 = 8 + r - (8 + r)/m - 2m \); but, by [3, (2)], \( K^2 = 6 \), \( 3 \), or 2, which are impossible. If \( m = 4 \), then \( r_p \leq 2 \), and, by (3)-(5), \( 3s + 4t + 7u = 24 \), and, by (7), \( r = 6 \) or 7 are the only solutions. Hence, by [3, (2)], \( r = 6 \), \( K^2 = 2 \), and \((a, b) = (4,4)\). If \( g(C) = 3 \) with \( p_a < 0 \) and \( Y \) is geometrically ruled, then, by [3, (2)], \( p_a = -1 \) and \((a, b) = (2,6) \) or \( p_a = -3 \) and \((a, b) = (4,4)\). Since \( Y \) is assumed geometrically ruled, we get from (3) and (4) that \( p_a = -1 \) implies \( 4 = 8 - 8/m \); hence, \( m = 2 \), \( e = -1 \), \( n = 1 \), and \( p_a = -3 \) imply \( 4 = 8 - 8/m + 4m \); hence, \( m = 1 \), \( e = 0 \), \( n = 4 \). If \( g(C) = 3 \) with \( p_a < 0 \) and \( Y \) not geometrically ruled, then, by III, \( m = 2 \) and, by (3)-(5), \( 8 + 8p_a = s + u \geq 0 \). Hence, \( p_a = -1 \) and \( Y \) is either of type (3, 5) with \( K^2 = -3 \), or of type (4, 4) with \( K^2 = -4 \). If \( g(C) > 3 \), then the proof in [3] for \( d = 8 \) applies.
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