NUMERICAL RADIUS-ATTAINING OPERATORS ON $C(K)$

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Abstract. Using a construction due to Johnson and Wolfe, we show that the numerical radius-attaining operators from $C(K)$ into itself are dense in the space of all operators, where $K$ is a compact Hausdorff space.

Let $X$ be a Banach space, $L(X)$ the Banach space of bounded linear operators from $X$ into $X$, and $NRA(X)$ the subset of $L(X)$ consisting of the numerical radius-attaining operators.

Berg and Sims [1] have proved the “Bishop-Phelps type” result that $NRA(X)$ is dense in $L(X)$ when $X$ is uniformly convex. Elsewhere we have shown the same to be so for $X$ being $c_0$, $l_1$, $L_1(\mu)$ or a uniformly smooth space.

In this note we consider the case of $X = C(K)$, the space of continuous real-valued functions on the compact Hausdorff space $K$. Following the lead of Johnson and Wolfe [3], we again show that $NRA(C(K))$ is dense in $L(C(K))$.

We still do not know of any $X$ for which $NRA(X)$ is not dense in $L(X)$. It may be that Lindenstrauss’s example, using a renorming of $c_0$, for which the norm-attaining operators are not dense in $L(X)$ [4] also serves in the present setup, but we have not yet found that to be so.

We introduce initially some definitions and notations.

We define the numerical radius of a bounded linear operator $T: X \to X$, denoted by $v(T)$, by

$$v(T) = \sup\{ |x^*(Tx)| : (x, x^*) \in \Pi(X) \},$$

where $\Pi(X) = \{ (x, x^*) \in X \times X^*: \|x^*\| = \|x\| = x^*(x) = 1 \}$.

We say that $T$ attains its numerical radius if there is $(x_0, x_0^*) \in \Pi(X)$ such that $v(T) = |x_0^*(Tx_0)|$, and we denote the set of numerical radius-attaining operators by $NRA(X)$.

If $K$ is a compact Hausdorff space and $X$ is a Banach space, we denote by $C_w(K, X^*)$ the Banach space of continuous functions $F: K \to X^*$, where $X^*$ is equipped with its $w^*$-topology, with the norm $\|F\| = \sup\{ \|F(t)\|: t \in K \}$.

It is a well-known result that $C_w(K, X^*)$ can be identified, isomorphically and isometrically, with the space $L(X, C(K))$ of all bounded linear operators from $X$ into $C(K)$, the identification being given by

$$(Tx)(t) = F(i)(x), \quad \forall t \in K, \forall x \in X,$$

where $T \in L(X, C(K))$ [2, p. 490].
$M(K)$ denotes the space of regular Borel measures on $K$, with the norm of the variation, and is identified with $C(K)^*$. In our case we will use the identification of $L(C(K))$ with $C_w^*(K, M(K))$.

For the proof of the result announced in the abstract we need several lemmas.

**Lemma 1.** Given $F \in C_w^*(K, M(K))$, $\epsilon > 0$, $f \in C(K)$, $t_0 \in K$ and an open set $V \subset K$, there is $U$, an open neighborhood of $t_0$, such that

(i) $|F(t)(V)| \geq |F(t_0)(V)| - \epsilon, \forall t \in U$;

(ii) $F(t)(f) \geq F(t_0)(f) - \epsilon, \forall t \in U$.

**Proof.** First we show that the function $\nu \in M(K) \mapsto |\nu(V)| \in \mathbb{R}$ is lower semicontinuous, where $M(K)$ has its $w^*$-topology.

In fact, if $\nu_0 \in M(K)$, by Hahn decomposition and regularity of $\nu_0$ we can choose disjoint compact sets $K^+$ and $K^-$, contained in $V$, such that $|\nu_0(K^+)| = \nu_0(K^+)$, $|\nu_0(K^-)| = -\nu_0(K^-)$ and $|\nu_0(V \setminus (K^+ \cup K^-)| < \epsilon/3$.

Since $K$ is compact Hausdorff, we can choose $f_0 \in C(K)$ with $|f_0(t)| \leq 1, \forall t \in K, f_0|_{K^+} = 1, f_0|_{K^-} = -1$ and $f_0|_{K \setminus V} = 0$.

Let $A = \{ \nu \in M(K): |\nu(f_0) - \nu_0(f_0)| < \epsilon/3 \}$. Then $A$ is a $w^*$-neighborhood of $\nu_0$, and if $\nu \in A$ we have

$$|\nu(V)| \geq \int_V f_0 \, d\nu \geq \left| \int_V f_0 \, d\nu_0 \right| = |\nu(f_0)| > |\nu_0(f_0)| - \frac{\epsilon}{3}$$

$$= \left| \int_V f_0 \, d\nu_0 \right| - \frac{\epsilon}{3} \geq \int_V f_0 \, d\nu_0 - \frac{\epsilon}{3}$$

$$= \int_{K^+} d\nu_0 - \int_{K^-} d\nu_0 + \int_{V \setminus (K^+ \cup K^-)} f_0 \, d\nu_0 - \frac{\epsilon}{3}$$

$$> \nu_0(K^+) - \nu_0(K^-) - \frac{\epsilon}{3} - \frac{\epsilon}{3} \geq \frac{\epsilon}{3}$$

Since $F \in C_w^*(K, M(K))$, the composite function $t \in K \mapsto |F(t)(V)|$ is also lower semicontinuous. Thus there is an open neighborhood $U_1$ of $t_0$ such that for $t \in U_1$ we have $|F(t)(V)| > |F(t_0)(V)| - \epsilon$.

Also, given $B = \{ \nu \in M(K): |\nu(f) - F(t_0)(f)| < \epsilon \}$, which is a $w^*$-neighborhood of $F(t_0) \in M(K)$, there is $U_2$, an open neighborhood of $t_0$ such that for $t \in U_2$ we have $F(t) \in B$, since $F \in C_w^*(K, M(K))$. Then if $t \in U_2$ we have $|F(t)(f)| \geq F(t_0)(f) - \epsilon$.

Letting $U = U_1 \cap U_2$ we have that $U$ is an open neighborhood of $t_0$ and, for $t \in U$, (i) and (ii) hold.

**Lemma 2.** Given $F \in C_w^*(K, M(K))$ and $\epsilon > 0$, there are $f_0 \in C(K)$, $\|f_0\|_{\infty} = 1$ and $t_0 \in K$ such that $F(t_0)(f_0) > \|F\| - \epsilon$ and $|F(t_0)(f)| = 1$.

**Proof.** Let $t_0 \in K$ be such that $|F(t_0)(K)| > \|F\| - \epsilon/3$.

For simplicity let us set $\mu_0 = F(t_0)$. Then $|\mu_0(K)| > \|F\| - \epsilon/3$.

Using Hahn decomposition and regularity of $\mu_0$, we can choose disjoint compact sets $K^+$ and $K^-$ such that $|\mu_0(K^+)| = \mu_0(K^+), |\mu_0(K^-)| = -\mu(K^-)$ and $|\mu_0(K \setminus (K^+ \cup K^-)| < \epsilon/3$.

Then $\mu_0(K^+) - \mu_0(K^-) > |\mu_0(K)| - \epsilon/3$. 


Case I. $t_0 \in K^+ \cup K^-$.  
Since $K$ is compact Hausdorff, we can choose $f_0 \in C(K)$, $|f_0(t)| \leq 1, \forall t \in K$, $f_0|_{K^+} = 1$, and $f_0|_{K^-} = -1$.  

Then
\[
F(t_0)(f_0) = \int_K f_0 \ d\mu_0 = \int_{K^+} d\mu_0 - \int_{K^-} d\mu_0 + \int_{K \setminus K^+ \cup K^-} f_0 \ d\mu_0 \\
= \mu_0(K^+) - \mu_0(K^-) + \int_{K \setminus K^+ \cup K^-} f_0 \ d\mu_0.
\]

Since
\[
\left| \int_{K \setminus K^+ \cup K^-} f_0 \ d\mu_0 \right| \leq |\mu_0|(K \setminus K^+ \cup K^-) < \epsilon/3,
\]
we get
\[
F(t_0)(f_0) > \mu_0(K^+) - \mu_0(K^-) - \epsilon/3 > |\mu_0|(K) - \epsilon/3 - \epsilon/3
\]
\[
> \|F\| - \epsilon/3 - \epsilon/3 - \epsilon/3 = \|F\| - \epsilon.
\]

Obviously in this case we have $|f_0(t_0)| = 1$, since $t_0 \in K^+ \cup K^-$ and $f_0|_{K^+ \cup K^-} = 1$.

Case II. $t_0 \not\in K^+ \cup K^-$.  
Since $K^+ \cup \{t_0\}$ and $K^-$ are again disjoint compact sets, let $f_0 \in C(K)$ be such that $|f_0(t)| \leq 1, \forall t \in K, f_0|_{K^+ \cup \{t_0\}} = 1$ and $f_0|_{K^-} = -1$.

As in Case I we have $F(t_0)(f_0) > \|F\| - \epsilon$ and $f_0(t_0) = 1$, by definition of $f_0$.

As an easy consequence we have

**Corollary 3.** $\nu(T) = \|T\|, \forall T \in C(K)$.  

The next lemma is a modification of a result of Johnson and Wolfe [3].

**Lemma 4.** Given $F \in C_{w^*}(K, M(K))$ and $\epsilon > 0$, there are open subsets $V_1$ and $V_2$ of $K$, with $\overline{V_1} \cap \overline{V_2} = \emptyset$, $V_2 \neq \emptyset$, and there are $f_1 \in C(K)$, $\|f_1\|_\infty = 1$, and $F_1 \in C_{w^*}(K, M(K))$ such that
\[
(i) |f_1(t)| = 1, \forall t \in K \setminus V_1;
(ii) |F_1(t)|(V_1) = 0, \forall t \in V_2;
(iii) F_1(t)(f_1) > \|F_1\| - \epsilon, \forall t \in V_2;
(iv) \|F - F_1\| < \epsilon.
\]

**Proof.** Let $t_0 \in K$ be such that $|F(t_0)(K)| > \|F\| - \epsilon/4$.

Using $(B^+, B^-)$ a Hahn decomposition of $K$ for $\mu_0 = F(t_0)$ and the regularity of $\mu_0$, choose $K^+ \subset B^+$ and $K^- \subset B^-$ compact sets such that
\[
\mu_0(K^+) - \mu_0(K^-) > |\mu_0|(K) - \epsilon/4 > \|F\| - \epsilon/2.
\]

As in the proof of Lemma 2, let $f_0 \in C(K)$, $\|f_0\|_\infty = 1$, be such that $f_0|_{K^+} = 1$, $f_0|_{K^-} = -1$ and $|f_0|_{K^+ \cup K^- \cup \{t_0\}} = 1$.

For each $\alpha \in [0, 1]$, let $A_\alpha = \{t \in K: |f_0(t)| < \alpha\}$.

Case I. $A_\alpha = \emptyset, \forall \alpha \in [0, 1]$.

In this case, $|f_0(t)| = 1, \forall t \in K$. Define $f_1 = f_0$, $V_1 = \emptyset$ and $F_1 = F$. Then (i) and (ii) hold for $t \in K$ and (iv) also is satisfied.
Moreover,

\[ F_1(t_0)(f_1) = F(t_0)(f_0) \geq \mu_0(K^+) - \mu_0(K^-) - \varepsilon/4 > \|F\| - 3\varepsilon/4. \]

By Lemma 1, using \( \varepsilon/4 \), there is \( V_2 \subset K \), an open neighborhood of \( t_0 \), such that

\[ F_1(t)(f_1) \geq F_1(t_0)(f_1) - \varepsilon/4, \forall t \in V_2. \]

Then \( F_1(t)(f_1) \geq \|F\| - \varepsilon = \|F_1\| - \varepsilon, \forall t \in V_2 \) and (iii) holds.

Obviously, \( V_2 \neq \emptyset \) and \( \overline{V_1} \cap \overline{V_2} = \emptyset \), and we are done.

**Case II.** There is \( \alpha_0 \in ]0,1[ \) with \( A_{\alpha_0} \neq \emptyset \).

In this case let \( \beta_0 \) be such that \( \alpha_0 < \beta_0 < 1 \).

Define \( V_1 = \{ t \in V : |f_0(t)| < \alpha_0 \} = A_{\alpha_0} \) and \( W = \{ t \in K : |f_0(t)| > \beta_0 \} \). Then \( V_1 \) and \( W \) are open sets, \( \overline{V_1} \cap W = \emptyset \) and \( \{t_0\} \cup K^+ \cup K^- \subset W \).

Since \( A_{\alpha_0} \neq \emptyset \), fix \( t_1 \in V_1 \) and choose \( f_1, g \in C(K), |f_1(t)| \leq 1, 0 \leq g(t) \leq 1, \forall t \in K \), such that

\[
\begin{align*}
\text{if } t \in (K \setminus V_1) \cap B^+, & \quad f_1(t) = 1, \\
\text{if } t \in (K \setminus V_1) \cap B^-, & \quad f_1(t) = -1, \\
\text{if } t = t_1, & \quad f_1(t) = 0,
\end{align*}
\]

and

\[
\begin{align*}
g(t) = \begin{cases} 0 & \text{if } t \in \overline{W}, \\ 1 & \text{if } t \in \overline{V_1}. \end{cases}
\end{align*}
\]

Then (i) holds and since

\[
\begin{align*}
[(1 - g)f_1](t) = 1 \quad & \text{if } t \in K^+, \\
[(1 - g)f_1](t) = -1 \quad & \text{if } t \in K^-,
\end{align*}
\]

we have

\[
F(t_0)((1 - g)f_1) = \int_K (1 - g)f_1 d\mu_0
\]

\[
= \int_{K^+} (1 - g)f_1 d\mu_0 + \int_{K^-} (1 - g)f_1 d\mu_0
\]

\[
+ \int_{(B^+ \setminus K^+) \cup (B^- \setminus K^-)} (1 - g)f_1 d\mu_0
\]

\[
\geq \mu_0(K^+) - \mu_0(K^-) - |\mu_0|((B^+ \setminus K^+) \cup (B^- \setminus K^-))
\]

\[
\geq |\mu_0|(K) - \varepsilon/4 - \varepsilon/4 > \|F\| - 3\varepsilon/4.
\]

By Lemma 1, using \( \varepsilon/4 \), there is \( U \subset K \) an open neighborhood of \( t_0 \) such that for each \( t \in U \),

\[
F(t)((1 - g)f_1) \geq F(t_0)((1 - g)f_1) - \varepsilon/4 > \|F\| - \varepsilon
\]

and

\[
|F(t)|(W) \geq |F(t_0)|(W) - \varepsilon/4 > \|F\| - \varepsilon.
\]

We can take \( U \cap \overline{V_1} = \emptyset \), since \( t_0 \notin \overline{V_1} \). Let \( V_2 \subset U \) be an open set such that \( t_0 \in V_2 \) and \( \overline{V_2} \subset U \). In particular, \( V_2 \neq \emptyset \) and \( \overline{V_1} \cap \overline{V_2} = \emptyset \).

Choose \( h \in C(K), \|h\|_{\infty} = 1, h(t) = 1 \) if \( t \in \overline{V_2} \) and \( h(t) = 0 \) if \( t \in K \setminus U \) and define \( F_1 : K \to M(K) \) by \( F_1(t) = [1 - h(t)g]F(t), \forall t \in K \), which means

\[
F_1(t)(p) = F(t)((1 - h(t)g)p), \quad \forall p \in C(K).
\]
Since \( g \in C(K) \), \( F_1(t) \in M(K) \), \( \forall t \in K \) and since \( h \in C(K) \) and \( F \in C_w(K, M(K)) \), \( F_1 \in C_w(K, M(K)) \). Also \( |F_1(t)|(K) \leq |F(t)|(K) \), since \( 1 - h(t)g \leq 1, \forall t \in K \), and then \( ||F_1|| \leq ||F|| \).

If \( t \in V_2 \), \( h(t) = 1 \) and \( F_1(t) = (1 - g)F(t) \). Since \( g|V_1 = 1, |F_1(t)|(V_1) = 0 \) and (ii) holds. Also

\[
F_1(t)(f_1) = F(t)((1 - g)f_1) > ||F|| - \varepsilon \geq ||F_1|| - \varepsilon
\]

and (iii) holds. For (iv), note that

\[
|F(t) - F_1(t)|(K) = |h(t)gF(t)|(K) = 0 \text{ if } t \in K \setminus U,
\]

since \( h|_{K \setminus U} = 0 \) and

\[
|F(t) - F_1(t)|(K) \leq |gF(t)|(K) \text{ if } t \in U.
\]

But \( g|_{\overline{W}} = 0 \) and then

\[
|gF(t)|(K) = |gF(t)|(K \setminus \overline{W}) \leq |F(t)|(K \setminus \overline{W}) = |F(t)|(K) - |F(t)|(\overline{W}) \leq ||F|| - |F(t)|(\overline{W}) < ||F|| - (||F|| - \varepsilon) = \varepsilon \text{ if } t \in U.
\]

Then

\[
||F - F_1|| = \sup\{|F(t) - F_1(t)|(K): t \in K\} \leq \varepsilon.
\]

The proof of the next lemma can be found in [3, Lemma 2.4].

**Lemma 5.** Let \( F \in C_w(K, M(K)) \), \( V_1, V_2 \subseteq K \) open sets, \( t_0 \in V_2 \), \( f_0 \in C(K) \), \( ||f_0||_{\infty} = 1 \), be such that

(a) \( |F(t)|(V_1) = 0, \forall t \in V_2 \);

(b) \( F(t_0)(f_0) \geq ||F|| - \varepsilon \);

(c) \( |f_0(t)| = 1, \forall t \in K \setminus V_1 \).

Then for every \( r > 2/3 \), there is \( F_1 \in C_w(K, M(K)) \), and there is \( t_1 \in V_2 \) such that

(i) \( |F_1(t)|(V_1) = 0, \forall t \in V_2 \);

(ii) \( F_1(t_1)(f_0) \geq ||F_1|| - re \);

(iii) \( ||F - F_1|| < re \).

**Theorem 6.** \( NRA(C(K)) = L(C(K)) \).

**Proof.** Let \( T \in L(C(K)) \) and \( \varepsilon > 0 \) be given, and let \( F \in C_w(K, M(K)) \) be the representative of \( T \).

Take \( 2/3 < r < 1 \) and apply Lemma 4 to get \( F_0 \in C_w(K, M(K)) \), \( V_1, V_2 \subseteq K \) open sets, \( \overline{V}_1 \cap \overline{V}_2 = \emptyset, V_2 \neq \emptyset \), \( f_0 \in C(K), ||f_0||_{\infty} = 1 \), such that

(a) \( |F_0(t)|(V_1) = 0, \forall t \in V_2 \);

(b) \( F_0(t)(f_0) > ||F_0|| - \varepsilon(1 - r), \forall t \in V_2 \);

(c) \( |f_0(t)| = 1, \forall t \in K \setminus V_1 \);

(d) \( ||F - F_0|| < \varepsilon(1 - r) \).

Choose \( t_0 \in V_2 \) such that

(b') \( F_0(t_0)(f_0) > ||F_0|| - \varepsilon(1 - r) \), and let \( \lambda = ||F_0|| - F_0(t_0)(f_0) \). Then \( 0 \leq \lambda < \varepsilon(1 - r) \).
Case I. \( \lambda = 0 \).

In this case, \( ||F_0|| = F_0(t_0)(f_0) = \delta_{t_0}(T_0f_0) \), where \( T_0 \in L(C(K)) \) corresponds to \( F_0 \).

Defining \( \mu_0 = (\text{sgn} f_0(t_0))\delta_{t_0} \), we have \( \mu_0(f_0) = |f_0(t_0)| = 1 \), since \( t_0 \in V_2 \) and \( V_1 \cap V_2 = \emptyset \), and \( |\mu_0(K)| = 1 \). Then \( (f_0, \mu_0) \in \Pi(C(K)) \).

Also we have \( T_0 \in \text{NRA}(C(K)) \), for

\[
|\mu_0(T_0f_0)| = |\delta_{t_0}(T_0f_0)| = ||F_0|| = ||T_0||.
\]

From (d), \( ||T - T_0|| = ||F - F_0|| < \epsilon(1 - r) < \epsilon \), and we are done.

Case II. \( \lambda > 0 \).

By definition of \( \lambda \),

\[
(b'') F_0(t_0)(f_0) = ||F_0|| - \lambda.
\]

Now (a), (b'') and (c) allow us to apply Lemma 5 and get \( F_1 \in C_w^{*}(K, M(K)) \) and \( t_1 \in V_2 \) such that

\[
(a_1) |F_1(t)(V_1)| = 0, \forall t \in V_2;
(b_1) F_1(t_1)(f_0) \geq ||F_1|| - r\lambda;
(d_1) ||F_0 - F_1|| < r\lambda.
\]

Again (a_1), (b_1) and (c) allow us to apply Lemma 5 and get \( F_2 \in C_w^{*}(K, M(K)) \) and \( t_2 \in V_2 \) such that

\[
(a_2) |F_2(t)(V_1)| = 0, \forall t \in V_2;
(b_2) F_2(t_2)(f_0) \geq ||F_2|| - r^2\lambda;
(d_2) ||F_1 - F_2|| < r^2\lambda.
\]

Following in this way we get sequences \( \{F_n\} \) in \( C_w^{*}(K, M(K)) \) and \( \{t_n\} \) in \( V_2 \) such that for each \( n \in \mathbb{N} \),

\[
(b_n) F_n(t_n)(f_0) \geq ||F_n|| - r^n\lambda;
(d_n) ||F_{n+1} - F_n|| \leq r^n\lambda.
\]

Since \( K \) is compact, \( \{t_n\} \) has a subsequence convergent to some \( \bar{t} \in K \). But \( t_n \in V_2, \forall n \in \mathbb{N} \) and then \( \bar{t} \in \overline{V_2} \). We still denote this subsequence by \( \{t_n\} \).

On the other hand, if \( m > n \geq 1 \), by (d_n) it follows that

\[
||F_n - F_m|| \leq \sum_{k=n+1}^{m} ||F_k - F_{k-1}|| \leq \left( \sum_{k=n+1}^{m} r^k \right) \lambda.
\]

Since \( r < 1 \), this shows that \( \{F_n\} \) is Cauchy in \( C_w^{*}(K, M(K)) \) and so is \( \{T_n\} \) in \( L(C(K)) \), where \( T_n \) corresponds to \( F_n, \forall n \in \mathbb{N} \).

Let \( \tilde{T} \in L(C(K)) \) be the limit of \( \{T_n\} \) and \( \tilde{F} \) its correspondent in \( C_w^{*}(K, M(K)) \).

We have

\[
||T - T_n|| \leq ||T - T_0|| + ||T_0 - T_n|| \leq \epsilon(1 - r) + \left( \sum_{k=1}^{n} r^k \right) \lambda
\]

\[
\leq \epsilon(1 - r) + \frac{r}{1 - r} \epsilon(1 - r) = \epsilon, \quad \forall n \in \mathbb{N}.
\]

From this it follows that \( ||T - \tilde{T}|| \leq \epsilon \).

It remains to show that \( \tilde{T} \in \text{NRA}(C(K)) \).
From $|\tilde{F}(t_n)(f_0) - F_n(t_n)(f_0)| < \|\tilde{F} - F_n\|$ and (b_n) we get
$$\tilde{F}(t_n)(f_0) > F_n(t_n)(f_0) - \|\tilde{F} - F_n\| \geq \|F_n\| - r^n\lambda - \|\tilde{F} - F_n\| \geq \|\tilde{F}\| - r^n\lambda - 2\|\tilde{F} - F_n\|, \quad \forall n \in \mathbb{N},$$
and since
$$\|\tilde{F} - F_n\| \leq \left( \sum_{k=n+1}^{\infty} r^k \right) \lambda = \frac{r^{n+1}}{1 - r} \lambda < r^{n+1}, \quad \forall n \in \mathbb{N},$$
we have
$$\tilde{F}(t_n)(f_0) > \|\tilde{F}\| - r^n\varepsilon(1 - r) - 2r^{n+1}\varepsilon = \|\tilde{F}\| - r^n\varepsilon(3 - r).$$

Now, since $\tilde{F}$ is w*-continuous and $t_n \to \tilde{t}$ and $r^n \to 0$, we have
$$\tilde{F}(\tilde{t})(f_0) \geq \|\tilde{F}\| \text{ or } \delta(\tilde{F}_0) \geq \|\tilde{T}\|.$$

Also $|f_0(\tilde{t})| = \lim_{n \to \infty} |f_0(t_n)| = 1$, since $f_0$ is continuous, $t_n \to \tilde{t}$ and $|f_0(t_n)| = 1$, because $t_n \in V_2$ and $\tilde{V}_2 \cap \tilde{V}_1 = \emptyset$.

Defining $\tilde{\mu} = (\text{sgn } f_0(\tilde{t}))\delta$, we have $\tilde{\mu} \in M(K)$, $|\tilde{\mu}|(K) = 1$, and $\tilde{\mu}(f_0) = |f_0(\tilde{t})| = 1$. Then $(f_0, \tilde{\mu}) \in \Pi(C(K)).$

Since $|\tilde{\mu}(\tilde{T}f_0)| = |\delta(\tilde{T}f_0)| \geq \|\tilde{T}\|$ and $\|\tilde{T}\| = \nu(\tilde{T})$, by Corollary 3, we get $|\tilde{\mu}(\tilde{T}f_0)| \geq \nu(\tilde{T})$. But $\nu(\tilde{T}) \geq |\tilde{\mu}(\tilde{T}f_0)|$, and then $\nu(\tilde{T}) = |\tilde{\mu}(\tilde{T}f_0)|$ and $\tilde{T} \in \text{NRA}(C(K))$.

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