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CONSTRUCTION OF A DIFFERENTIAL EQUATION $y'' + Ay = 0$ WITH SOLUTIONS HAVING THE PRESCRIBED ZEROS

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ABSTRACT. We show that an entire function $A(z)$ can be constructed so that the differential equation $y'' + Ay = 0$ has two linearly independent solutions having the prescribed points as their only zeros.

The main purpose of this brief note is to prove

THEOREM 1. *Let $\{a_n\}$ and $\{b_n\}$ be two given sequences with no finite limit point. If the two sequences have no points in common, that is $a_n \neq b_m$ for any n and m $(n, m = 1, 2, 3, \ldots)$, then there exists an entire function $A(z)$ such that the differential equation $y'' + Ay = 0$ has two linearly independent solutions w_1 and w_2 whose only zeros are $\{a_n\}$ and $\{b_n\}$, respectively.*

1. Preliminaries. Consider the differential equation (D.E.)

$$y'' + A(z)y = 0, \tag{1.1}$$

where A is entire. It is well known that

(i) all the solution of (1.1) are entire;

(ii) if w_1 and w_2 are two linearly independent solutions, then the Wronskian $W(z; w_1, w_2) = w_1 w_2' - w_1' w_2$ is constant, with no loss of generality, we assume that this constant is 1;

(iii) if z_0 is a zero of a solution, then its multiplicity is always equal to 1.

We say that *an entire function f has the BL property if, for each one of its zeros, say a, we have either $f'(a) = 1$ or $f'(a) = -1$.*

Set

$$f(z) = w_1(z) \cdot w_2(z). \tag{1.2}$$

It is easy to derive that f satisfies the D.E.

$$-4Af^2 = 1 - (f')^2 + 2ff''. \tag{1.3}$$

From (1.3) we readily conclude

LEMMA 1.1. *Let w_1 and w_2 be two linearly independent solutions of* (1.1). *Then $f = w_1 w_2$ has the BL property.*

It is an elementary exercise to show that *if an entire function f has the BL property, then the function A defined by*

$$-4A(z) = (1/f^2) - (f'/f)^2 + 2(f''/f) \tag{1.4}$$

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is entire. With this fact in mind, we now establish

LEMMA 1.2. *Let f be an entire function with the BL property. Then $f = w_1 w_2$, where w_1 and w_2 are some linearly independent solutions of D.E. with A defined by* (1.4).

PROOF. Let z_0 be a point such that $f(z_0) \neq 0$. Then we can choose a disk $D = \{z: |z - z_0| < r_0\}$ with the property that $f(z) \neq 0$ for all $z \in D$. In D, we define (by choosing a branch)

$$w_1(z) = (f)^{1/2} \exp\left(-\frac{1}{2}\int_{z_0}^{z} \frac{1}{f(s)}\, ds\right), \tag{1.5}$$

and

$$w_2(z) = (f)^{1/2} \exp\left(\frac{1}{2}\int_{z_0}^{z} \frac{1}{f(s)}\, ds\right). \tag{1.6}$$

Although it is not clear at the outset that w_1 and w_2 can be analytically continued uniquely to the entire complex plane, by a straightforward substitution, however, it can be easily shown that w_1 and w_2 both satisfy the D.E. (1.1) with A defined by (1.4). Thus, from (i), we conclude that w_1 and w_2 are both entire and $f = w_1 w_2$.

From Lemmas 1.1 and 1.2, we conclude

COROLLARY 1.3. *An entire function f has the BL property iff f is a product of two linearly independent solutions of* D.E. (1.1).

2. Proof of Theorem 1. Choose an entire function $g(z)$ so that its only zeros are $\{a_n\} \cup \{b_n\}$ and the multiplicity of each zero is one. We also choose an entire function h such that

$$\exp(h(z)) = \begin{cases} -1/g'(a_n) & \text{if } z = a_n, \\ 1/g'(b_n) & \text{if } z = b_n. \end{cases} \tag{2.1}$$

The algorithms to construct g and h are well known; however, such g and h are not unique [**1**, pp. 295 and 298].

Define

$$f = g \exp(h). \tag{2.2}$$

Then, from (2.1), f has the BL property and

$$f'(z) = \begin{cases} -1 & \text{if } z = a_n, \\ 1 & \text{if } z = b_n. \end{cases} \tag{2.3}$$

Thus, from Lemma 1.2, there exists an entire function A and $f = w_1 w_2$, where w_1 and w_2 are two linearly independent solutions of the D.E. $y'' + A(z)y = 0$ defined by (1.5) and (1.6) respectively. We now show that w_1 and w_2 have the desired property.

Let

$$F(z) = \exp\left(\int^{z} 1/f(s)\, ds\right). \tag{2.4}$$

From (1.5), $F(z) = f/w_1^2$. Therefore, $F(z)$ is meromorphic. From (2.3), we see that in a small neighborhood U of $z = a_n$

$$1/f(z) = -1/(z - a_n) + H_n(z), \tag{2.5}$$

where H_n is holomorphic in U. From (2.4) and (2.5),

$$F'/F = 1/f = -1/(z - a_n) + H_n(z) \qquad (z \in U). \tag{2.6}$$

Therefore, (2.6) implies that F has a simple pole at $z = a_n$. A similar argument shows that F has a simple zero at $z = b_n$. Since the only zeros of f are $\{a_n\} \cup \{b_n\}$, hence $\{a_n\}$ and $\{b_n\}$ are the only poles and zeros of F, respectively. This immediately implies that F' $(= F/f)$ has no zeros and has double poles at $\{a_n\}$. Thus $w_1 = 1/(F')^{1/2}$ is an entire function whose zeros are precisely $\{a_n\}$. Since $w_2 = w_1 F$, the only zeros of w_2 are $\{b_n\}$. This completes the proof.

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