

REMARK ON THE GENERALIZED PUTNAM–FUGLEDE THEOREM

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ABSTRACT. If A^* is M -hyponormal, B is dominant, and if $CA = BC$, then $CA^* = B^*C$.

Let \mathcal{H} be a Hilbert space. An operator $A \in \mathcal{B}(\mathcal{H})$ is dominant if $\text{range}(A - \lambda I) \subset \text{range}(A - \lambda I)^*$ for all $\lambda \in \sigma(A)$, or, equivalently, for all $\lambda \in \mathbb{C}$ because

$$\text{range}(A - \lambda I) = \text{range}(A - \lambda I)^* = \mathcal{H}$$

for $\lambda \notin \sigma(A)$. By the result of Douglas [1], it is equivalent to the existence of a real number M_λ for each $\lambda \in \mathbb{C}$ such that

$$\|(A - \lambda I)^*x\| \leq M_\lambda \|(A - \lambda I)x\|$$

for all $x \in \mathcal{H}$. If there is a constant M such that $M_\lambda \leq M$ for all $\lambda \in \mathbb{C}$, A is called M -hyponormal, and if $M = 1$, A is hyponormal.

Stampfli and Wadhwa [5] showed that if $A^* \in \mathcal{B}(\mathcal{H})$ is hyponormal, $B \in \mathcal{B}(\mathcal{H})$ is dominant, $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is injective and has dense range, and if $CA = BC$, then A and B are normal.

Radjabalipour [3] improved this result by allowing A^* to be M -hyponormal. Of course, the condition that A and B are normal allows one to conclude immediately by the usual Putnam–Fuglede theorem that $CA^* = B^*C$.

Moore, Rogers and Trent [2] showed that, if A^* and B are both M -hyponormal, the conclusion $CA^* = B^*C$ can be reached with no restrictions on C at all.

We show that this result can also be improved by allowing B to be dominant.

We need the following two lemmas:

LEMMA 1 [4]. *Let A be dominant and let \mathcal{M} be an invariant subspace of A for which $A|_{\mathcal{M}}$ is normal. Then \mathcal{M} reduces A .*

LEMMA 2. *The restriction $A|_{\mathcal{M}}$ of the dominant (respectively, M -hyponormal) operator A on \mathcal{H} to an invariant subspace \mathcal{M} of A is dominant (respectively, M -hyponormal).*

PROOF. Let $P_{\mathcal{M}}$ be the orthogonal projection from \mathcal{H} to \mathcal{M} . Then

$$\begin{aligned} \|(A|_{\mathcal{M}} - \lambda I)^*x\| &= \|P_{\mathcal{M}}(A - \lambda I)^*x\| \leq \|(A - \lambda I)^*x\| \\ &\leq M_\lambda \text{ (resp. } M) \|(A - \lambda I)x\| = M_\lambda \text{ (resp. } M) \|(A|_{\mathcal{M}} - \lambda I)x\| \end{aligned}$$

for each $\lambda \in \mathbb{C}$ and all $x \in \mathcal{M}$.

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THEOREM. *If $A^* \in \mathcal{B}(\mathcal{H})$ is M -hyponormal, $B \in \mathcal{B}(\mathcal{K})$ is dominant, and if $CA = BC$ for $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then $CA^* = B^*C$.*

PROOF. From $CA = BC$ we know that $(\ker C)^\perp$ and $(\text{range } C)^\perp$ are invariant subspaces of A^* and B , respectively. Hence $A^*(\ker C)^\perp$ is M -hyponormal and $B(\text{range } C)^\perp$ is also dominant by Lemma 2. By the decompositions $\mathcal{H} = (\ker C)^\perp \oplus (\ker C)$ and $\mathcal{K} = (\text{range } C)^\perp \oplus \{(\text{range } C)^\perp\}^\perp$, we have

$$A = \begin{pmatrix} A_1 & 0 \\ * & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & * \\ 0 & B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here $A_1^* = A^*(\ker C)^\perp$ is M -hyponormal, $B_1 = B(\text{range } C)^\perp$ is dominant and C_1 is injective with dense range. We obtain $C_1 A_1 = B_1 C_1$ from $CA = BC$. Hence, A_1 and B_1 are normal by [3, Theorem 3(a)] and $C_1 A_1^* = B_1^* C_1$ by the usual Putnam–Fuglede theorem. Then, by Lemma 1, $(\ker C)^\perp$ and $(\text{range } C)^\perp$ reduces A^* and B to normal operators, respectively. Therefore, we have

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}.$$

Hence we obtain $CA^* = B^*C$.

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