PROPERTY (H) IN LEBESGUE-BOCHNER FUNCTION SPACES

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Abstract. We prove that if a Banach space $X$ has the property (HR) and if $l_1$ is not isomorphic to a subspace of $X$, then every point on the unit sphere of $X$ is a denting point of the closed unit ball. We also prove that if $X$ has the above property, then $L^p(\mu, X)$, $1 < p < \infty$, has the property (H).

1. Introduction. A Banach space is said to have the property (H) [1] (known also as the Radon-Riesz property [10] or the Kadec-Klee property [3]) if every sequence of norm-one elements that converges weakly to a norm-one element converges in norm. M. A. Smith and B. Turett [10] proved that if $(\Omega, \Sigma, \mu)$ is a measure space that is not purely atomic and $L^p(\mu, X)$, $1 < p < \infty$, has the property (H), then $X$ is strictly convex (R). They posed the following problem:

Question. If $X$ is a strictly convex Banach space with property (H), does $L^p(\mu, X)$, $1 < p < \infty$, have property (H)?

The purpose of this paper is to show that the answer to the above question is affirmative when $l_1$ is not isomorphic to a subspace of $X$. Let us recall some other definitions.

Every element in the unit sphere is a denting point of the closed unit ball, i.e., if $\|x_0\| = 1$ then $x_0 \not\in \text{co}(M(x_0, \varepsilon))$ for all $\varepsilon > 0$ where $M(x_0, \varepsilon) = \{x \in X : \|x\| \leq 1 \text{ and } \|x - x_0\| \geq \varepsilon\}$.

If, for each $\varepsilon > 0$ and $z \in X$ with $\|z\| = 1$, there exists $\delta(z, \varepsilon) > 0$ such that if $x$ and $y$ are in $X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$ then $\|x + y - 2z\| \geq \delta(z, \varepsilon)$.

The properties (G) and (HR) were introduced by Ky Fan and I. Glicksberg [2]. They proved that (G) implies (HR) and that (G) and (HR) are equivalent when $X$ is reflexive. Midpoint local uniform convexity (MLUR) was introduced by K. W. Anderson (see [10]). It is easy to see that it is equivalent to the following property:

If, for each $\varepsilon > 0$, $0 < \alpha \leq \beta < 1$ and $z \in X$ with $\|z\| = 1$, there exists $\delta(z, \varepsilon, \alpha, \beta) > 0$ such that if $x$ and $y$ are in $X$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|y - x\| > \varepsilon$ then $\|\nu x + (1 - \nu)y - z\| \geq \delta(z, \varepsilon, \alpha, \beta)$ when $\alpha \leq \nu \leq \beta$.

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It is known that if $X$ has the property (G), then $X$ is (MLUR). On the other hand, M. A. Smith [9] renormed $l_1$ such that it has the property (HR), but it is not (MLUR). M. I. Kadec [4] used H. P. Rosenthal’s $l_1$ theorem to show that if $X$ has the property (HR) and if $X$ does not contain $l_1$, then $X$ is (MLUR). We prove the following two theorems.

**Theorem 1.** If $X$ has the property (HR), and if $l_1$ is not isomorphic to a subspace of $X$, then $X$ has the property (G).

**Theorem 2.** Suppose $X$ has the property (G). Then $L^p(\mu, X), 1 < p < \infty$, has the property (H).

For more geometrical properties between $X$ and $L^p(\mu, X)$, we suggest the reader consult [6 and 10]. We wish to thank Professor B. Turett for informing us of the paper of Kadec [4].

2. **Proof of Theorem 1.** We may assume that $X$ is separable. Suppose $X$ does not have the property (G). Then there are $x_0$ with $\|x_0\| = 1$ and $\varepsilon > 0$ such that $x_0 \in \overline{\text{co}}M(x_0, \varepsilon)$ where $M(x_0, \varepsilon) = \{x \in X: \|x - x_0\| \geq \varepsilon \text{ and } \|x\| \leq 1\}$. If $x_0$ belongs to the weak closure of $M(x_0, \varepsilon)$, since $l_1$ is not isomorphic to a subspace of $X$, then there is a sequence $(x_n)$ in $M(x_0, \varepsilon)$ which converges to $x_0$ weakly [5]. Since $X$ has the property (H), $(x_n)$ converges to $x_0$ in norm. This contradicts the fact $\|x_n - x_0\| \geq \varepsilon$ for all $n$. So there exist $\delta > 0$ and $f_1, f_2, \ldots, f_n$ in $X^*$ such that, if $x \in M(x_0, \varepsilon)$, then $f_k(x) < f_k(x_0) - \delta$ for some $k \leq n$. Let

$$M_k(x_0, \varepsilon) = \{x: x \in M(x_0, \varepsilon) \text{ and } f_k(x) < f_k(x_0) - \delta\}.$$  

If $x_0 \notin \overline{\text{co}}(M_1(x_0, \varepsilon) \cup M_2(x_0, \varepsilon))$, then there exist $\delta' > 0$ and $f \in X^*$ such that if $x \in M_1(x_0, \varepsilon) \cup M_2(x_0, \varepsilon)$ then $f(x) < f(x_0) - \delta'$. Hence, in this case, $M(x_0, \varepsilon) = \bigcup_{k=3}^n M_k(x_0, \varepsilon) \cup M_f(x_0, \varepsilon)$ where $M_f(x_0, \varepsilon) = \{x: x \in M(x_0, \varepsilon), f(x) < f(x_0) - \delta'\}$. So we may assume that

$$x_0 \in \overline{\text{co}}M(x_0, \varepsilon) = \overline{\text{co}}(M_1(x_0, \varepsilon) \cup M_2(x_0, \varepsilon)).$$

and there exist sequences $(x_n)$ in $M_1(x_0, \varepsilon)$, $(y_n)$ in $M_2(x_0, \varepsilon)$ and $(\alpha_n)$ with $0 \leq \alpha_n \leq 1$ such that $\lim_{n \to \infty} \alpha_n x_n + (1 - \alpha_n) y_n = x_0$. Obviously, $0 < \lim \alpha_n < 1$. This contradicts the fact that $X$ is (MLUR). So $X$ has the property (G). Q.E.D.

3. **Proof of Theorem 2.** Let $(f_n)$ be a norm-one sequence in $L^p(\mu, X)$ which converges weakly to a norm-one element $f$. Let $g$ be an element in $L^p(\mu, X)^*$ such that $g(f) = 1 = \|g\|$. Then

$$1 \geq \frac{1}{2} \|f_n(\cdot)\| + \|f(\cdot)\|_{L^p(\mu)} \geq \frac{1}{2} \|f_n + f\|_{L^p(\mu, X)} \geq \frac{1}{2} g(f_n + f).$$

Since this last term converges to 1 and $L^p(\mu), 1 < p < \infty$, is uniformly convex, $\|f_n(\cdot)\|$ converges to $\|f(\cdot)\|$ in $L^p(\mu)$. By passing to subsequence and perturbing the sequence $(f_n)$, we may assume that $\|f_n(t)\| = \|f(t)\|$ for all $n \in \mathbb{N}$ and $t \in \Omega$. Let

$$d(n, k) = \left\{ t: \|f_n(t) - f(t)\| > \|f(t)\|/k \right\}.$$
We claim that if \( \lim_{n \to \infty} \int_{d(n, k)} \|f(t)\|^p \, d\mu = 0 \) for all \( k \in \mathbb{N} \), then \((f_n)\) converges to \( f \) in norm. Given \( \varepsilon > 0 \), there is a \( k \in \mathbb{N} \) such that \( 1/k < \varepsilon/2 \). Since \( \lim_{n \to \infty} \int_{d(n, k)} \|f(t)\|^p \, d\mu = 0 \), there exists \( N \) such that, if \( n > N \), then
\[
\int_{d(n, k)} \|f(t)\|^p \, d\mu < \varepsilon^p/4^p.
\]
Hence, if \( n > N \), then
\[
\int \|f(t) - f_n(t)\|^p \, d\mu
\]
\[
= \int_{d(n, k)} \|f(t) - f_n(t)\|^p \, d\mu + \int_{\Omega - d(n, k)} \|f(t) - f_n(t)\|^p \, d\mu
\]
\[
< 2^p \int_{d(n, k)} \|f(t)\|^p \, d\mu + \int_{\Omega - d(n, k)} \|f(t)\|^p/k^p \, d\mu
\]
\[
< \frac{\varepsilon^p}{2^p} + \frac{\varepsilon^p}{2^p} < \varepsilon^p.
\]
Now, suppose that \((f_n)\) does not converge to \( f \) in norm; then there exist \( \varepsilon > 0 \) and \( k \in \mathbb{N} \) such that
\[
\int_{d(n, k)} \|f(t)\|^p \, d\mu > \varepsilon \quad \text{for infinitely many \( n \).}
\]
By passing to a subsequence, we may assume that
\[
\int_{d(n, k)} \|f(t)\|^p \, d\mu > \varepsilon \quad \text{for all \( n \).}
\]
Since \((f_n)\) converges to \( f \) weakly, for each \( m \in \mathbb{N} \), there is \((a_i^m)_{i=1}^\infty \) such that \( \Sigma_{i=1}^\infty a_i^m = 1 \), \( a_i^m > 0 \) and \( \|\Sigma_{i=1}^\infty a_i^m f_i - f\| < 1/m \). By passing to a sequence of \((\Sigma_{i=1}^\infty a_i^m f_i)_{m=1}^\infty \), we may assume that \((\Sigma_{i=1}^\infty a_i^m f_i)\) converges to \( f \) a.e. Let \( \lambda \) be the probability given by
\[
\lambda(A) = \int_A \|f(t)\|^p \, d\mu \quad \text{for all \( A \in \Sigma \).}
\]
Then \( \lambda(d(n, k)) > \varepsilon \). For each \( m \), let
\[
S(m) = \left\{ t: \left( \sum_{i=1}^N a_i^m x_{d(i, k)}(t) \right)(t) > \frac{\varepsilon}{2} \right\}.
\]
Since \( \sum_{i=1}^N a_i^m x_{d(i, k)}(t) \, d\lambda > \varepsilon \) and \( \lambda \) is a probability measure, \( \lambda(S(m)) > \varepsilon/2 \). Let \( S = \{ t: t \in d(m, k) \text{ for infinitely many } m \} \). Then \( \lambda(S) \neq 0 \) and there is \( t' \in S \) such that \( (\Sigma_{i=1}^N a_i^m f_i(t'))_{m=1}^\infty \) converges to \( f(t') \) in norm. Let \( T = \{ n: t' \in d(n, k) \} \). If \( t' \in S(m) \), then \( \Sigma_{i \in T} a_i^m > \varepsilon/2 \) and
\[
\sum_{i=1}^N a_i^m f_i(t') = \sum_{i \in T} a_i^m f_i(t') + \sum_{i \notin T} a_i^m f_i(t')
\]
\[
= \left( \sum_{i \in T} a_i^m \right) \left( \sum_{i \in T} a_i^m f_i(t') / \sum_{i \in T} a_i^m \right)
\]
\[
+ \left( \sum_{i \notin T} a_i^m \right) \left( \sum_{i \notin T} a_i^m f_i(t') / \sum_{i \notin T} a_i^m \right).
Since \( \| f_i(t') - f(t') \| > \| f(t') \| /k, \| f_i(t') \| = \| f(t') \| \) for all \( i \in T \) and \( X \) has the property (G), there is \( \delta > 0 \) such that if \( x \in \text{co}\{ f_i(t') | i \in T \} \) then \( \| x - f(t') \| > \delta \). Since \( t' \in S(m) \) infinitely often, there exist sequences \( (x_n), (y_n) \) and \( (\alpha_n) \) such that \( \| x_n - f(t') \| > \delta \) for all \( n, 1 \geq \alpha_n > \epsilon /2 \) and

\[
\lim_{n \to \infty} \alpha_n x_n + (1 - \alpha_n) y_n = f(t').
\]

This contradicts the fact that \( X \) is (MLUR). So \( f_n \) must converge to \( f \) in norm.

Q.E.D.

4. Since a Banach space \( X \) contains a copy of \( l_1 \) if and only if \( L^p(\mu, X), 1 < p < \infty \), contains a copy of \( l_1 \) [7], we have the following theorem.

**Theorem 3.** If \( l_1 \) is not isomorphic to a subspace of \( X \), then the following are equivalent:

(i) \( X \) has the property (HR),
(ii) \( X \) has the property (G),
(iii) \( L^p(\mu, X), 1 < p < \infty \), has the property (HR),
(iv) \( L^p(\mu, X), 1 < p < \infty \), has the property (G).

**References**


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