AFFINE ENDOMORPHISMS WITH A DENSE ORBIT

TATSURO KASUGA

ABSTRACT. For a continuous endomorphism f on a locally compact group X and $a \in X$, we define an affine endomorphism $f_a \colon X \to X$. We prove that if f_a is not one to one and if (X, f_a) has a dense orbit then X is compact.

Let X be a locally compact group and let $f_a \colon X \to X$ be an affine endomorphism: i.e. $f_a(x) = af(x)$ where a is a fixed element of X and f is a continuous endomorphism of X onto itself. The following problem is raised by P. R. Halmos [4, p. 29]. "Can an automorphism of a locally compact but noncompact group be an ergodic measure-preserving transformation?" Recently the problem of Halmos was solved by N. Aoki [1]. Halmos' problem was generalized by Dani and Dateyama-Kasuga who obtained the following:

THEOREM A. *Let X be a locally compact metric group and $f_a \colon X \to X$ be a bicontinuous affine automorphism. If (X, f_a) has a dense orbit, then*

(i) *X is compact when X is connected (S. G. Dani [2]).*

(ii) *X is either compact or discrete when X is totally disconnected [3].*

The aim of this note is to investigate whether the problem holds for continuous affine endomorphisms. The following is the main result of this note.

THEOREM 1. *Let X be a locally compact group and let $f_a \colon X \to X$ be a continuous affine endomorphism that is not one-to-one. If (X, f_a) has a dense orbit then X is compact.*

We will deduce Theorem 1 from the following consequence of Theorem A.

THEOREM B. *Let X be a locally compact metric group and $f_a \colon X \to X$ be a bicontinuous affine automorphism. If X is not discrete and (X, f_a) has a dense orbit, then X is compact.*

PROOF. Let X_0 be the connected component of the identity e in X. Then X/X_0 is a locally compact totally disconnected metric group. Now define the map $g_a \colon X/X_0 \to X/X_0$ by

$$g_a(xX_0) = f_a(x)X_0 \qquad (x \in X).$$

Then it is obvious that g_a is a bicontinuous affine automorphism and $(X/X_0, g_a)$ has a dense orbit. By Theorem A, X/X_0 is compact. Therefore it is enough to show that X_0 is compact. This is proved by using the fact that an affine automorphism

---

with a dense orbit is measure-preserving under the left Haar measure μ (cf. [**3**, Proposition 1]).

The conclusion of Theorem 1 will be obtained in proving the following two propositions:

PROPOSITION 2. *Let X and f_a be as in Theorem 1. If X is discrete and (X, f_a) has a dense orbit, then X is finite.*

PROOF. Since f_a is not one-to-one, there are z_1 and z_2 in X such that $z_1 \neq z_2$, but $f_a(z_1) = f_a(z_2)$. Since (X, f_a) has a dense orbit, there is an element x_0 in X such that $\{f_a^n(x_0): n \geq 0\}$ is dense in X. Hence there are integers n_1 and n_2 $(0 < n_1 < n_2)$ such that $f_a^{n_1}(x_0) = z_1$ and $f_a^{n_2}(x_0) = z_2$. Since $f_a^{n_1+1}(x_0) = f_a(z_1) = f_a(z_2) = f_a^{n_2+1}(x_0)$, we have

$$X = \left\{ x_0, f_a(x_0), f_a^2(x_0), \ldots, f_a^{n_2-n_1-1}(x_0) \right\},$$

i.e. X is finite.

For the case X is not discrete, X is σ-compact when (X, f_a) has a dense orbit. For, since X is locally compact, there exists a compact neighborhood V of e in X. By assumption, there is $x_0 \in X$ such that $\{f_a^n(x_0): n \geq 0\}$ is dense in X. Hence, $X = \bigcup_{n=0}^{\infty} f_a^n(x_0)V$, i.e. X is σ-compact. Clearly, f_a is an open map, because $f: X \to X$ is onto.

Hereafter we may assume that X is σ-compact. Then there exists a compact normal subgroup H of X such that $f(H) \subset H$ and that X/H is metrizable and separable (see [**1**]). Define the endomorphism $g: X/H \to X/H$ by $g(xH) = f(x)H$ and put $g_a(xH) = ag(xH)$ $(x \in X)$. If (X, f_a) has a dense orbit, then so does $(X/H, g_a)$.

PROPOSITION 3. *Let X be a locally compact metric group with a left invariant metric d and let $f_a: X \to X$ be a continuous affine endomorphism. If X is not discrete and if (X, f_a) has a dense orbit, then X is compact.*

PROOF. Define a metric d_0 by

$$d_0(x, y) = \frac{d(x, y)}{d(x, y) + 1} \qquad (x, y \in X).$$

Then d_0 is bounded and is equivalent to d. We let

$$\overline{X} = \left\{ (x_i)_{i=0}^{\infty}: f(x_{i+1}) = x_i, i \geq 0 \right\}.$$

Put $\bar{f}((x_i)_{i=0}^{\infty}) = (f(x_i))_{i=0}^{\infty}$ and $\varphi_0((x_i)_{i=0}^{\infty}) = x_0$ for $(x_i)_{i=0}^{\infty} \in \overline{X}$. Then $\bar{f}$ is a continuous group automorphism and $\varphi_0 \circ \bar{f} = f \circ \varphi_0$ holds. Define the metric for $\overline{X}$ by

$$\bar{d}(\bar{x}, \bar{y}) = \sum_{i=0}^{\infty} d_0(x_i, y_i)/2^i \qquad \left(\bar{x} = (x_i)_{i=0}^{\infty}, \bar{y} = (y_i)_{i=0}^{\infty} \in \overline{X}\right).$$

Clearly $\overline{X}$ is a locally compact metric group and $\bar{d}$ is rotation invariant since d_0 is rotation invariant. For $\bar{a} = (a_i)_{i=0}^{\infty} \in \overline{X}$ with $a_0 = a$, and affine endomorphism $\bar{f}_{\bar{a}}$ has a dense orbit in $\overline{X}$ since so does (X, f_a). Indeed, we can find $z_0 \in X$ such that

$\{f_a^n(z_0); n \geqslant 0\}$ is dense in X. We fix the point z_0. Since f is onto, there is $\bar{y} = (y_i)_{i=0}^{\infty} \in \overline{X}$ with $y_0 = z_0$ such that $f(y_{i+1}) = y_i$ for $i \geqslant 0$. Let $\varepsilon > 0$. For $\bar{x} = (x_i)_{i=0}^{\infty} \in \overline{X}$, there is $m > 0$ such that

$$\sum_{i=m+1}^{\infty} d_0(x_i, y_i)/2^i < \varepsilon/2,$$

and so we choose $\eta > 0$ such that $d_0(f^j(x), f^j(y)) < \varepsilon/4 \ (0 \leqslant j \leqslant m)$ when $d_0(x, y) < \eta$. Take $\bar{a} = (a_i)_{i=0}^{\infty} \in \overline{X}$ such that $a_0 = a$ and $f^j(a_i) = a_{j-i} \ (0 \leqslant i \leqslant j \leqslant m)$. Since $\{f_a^n(z_0); n \geqslant 0\}$ is dense in X, for $a_1^{-1}a_2^{-1} \cdots a_m^{-1}x_m \in X$, there is $k > 0$ such that

$$d_0(f_a^k(z_0), a_1^{-1}a_2^{-1} \cdots a_m^{-1}x_m) < \eta.$$

Hence

$$d_0\Big(a_{m-j}a_{m-j-1} \cdots a_2 a_1 f_a^{k+j}(z_0), f^j(x_m)\Big)$$
$$= d_0\Big(a_{m-j}a_{m-j-1} \cdots a_2 a_1 a_0 f(a_0) \cdots f^{j-1}(a_0) f^j(f_a^k(z_0)), f^j(x_m)\Big)$$
$$= d_0\Big(f^j(f_a^k(z_0)), f^j(a_1^{-1})f^j(a_2^{-1}) \cdots f^j(a_m^{-1})f^j(x_m)\Big) < \varepsilon/4$$

for $0 \leqslant j \leqslant m$. Therefore,

$$\bar{d}\big(\bar{f}_{\bar{a}}^{k+m-1}(\bar{y}), \bar{x}\big) \leqslant \sum_{j=0}^{m} d_0\Big(a_{m-j}a_{m-j-1} \cdots a_2 a_1 f_a^{k+j}(z_0), f^j(x_m)\Big)/2^j + \varepsilon/2$$
$$< \varepsilon,$$

which implies that $(\overline{X}, \bar{f}_{\bar{a}})$ has a dense orbit.

Obviously $\overline{X}$ is locally compact and $\bar{f}_{\bar{a}}: \overline{X} \to \overline{X}$ is one-to-one. Since $\bar{f}_{\bar{a}}$ is an open map, actually $\bar{f}_{\bar{a}}$ is bicontinuous. Since $(\overline{X}, \bar{f}_{\bar{a}})$ has a dense orbit and $\overline{X}$ is not discrete, $\overline{X}$ is compact by Theorem B. Hence $\varphi_0(\overline{X}) = X$ is compact. The proof is completed.

References

1. N. Aoki, *Dense orbits of automorphisms and compactness of groups*, Topology Appl. (to appear).

2. S. G. Dani, *Dense orbits of affine automorphisms and compactness of groups*, J. London Math. Soc. (2) **25** (1982), 241–245.

3. M. Dateyama and T. Kasuga, *Ergodic affine maps of locally compact groups*, J. Math. Soc. Japan (1985), 363–372.

4. P. R. Halmos, *Lectures on ergodic theory*, Publ. Math. Soc. Japan, Tokyo, 1956.

5. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, New York, 1955.

DEPARTMENT OF MATHEMATICS, KUMAMOTO RADIO TECHNICAL COLLEGE, NISHIGOSHI-MACHI, KIKUCHI-GUN, KUMAMOTO, 861-11, JAPAN