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THE MULTIPLICITY OF THE STEINBERG REPRESENTATION OF $\mathrm{GL}_n \mathbf{F}_q$ IN THE SYMMETRIC ALGEBRA

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ABSTRACT. Let $S(V)$ denote the symmetric algebra on the standard n-dimensional representation V of $\mathrm{GL}_n \mathbf{F}_q$. The *multiplicity series* in $S(V)$ for the Steinberg representation St of $\mathrm{GL}_n \mathbf{F}_q$ is determined. This series is defined by $F_{\mathrm{St}}(t) = \sum_{k=0}^{\infty} a_k t^k$, where a_k is the multiplicity of St in the kth symmetric power $S^k(V)$. We show that $F_{\mathrm{St}}(t) = t^r \prod_{i=1}^{n}(1 - t^{q^i - 1})^{-1}$, where $r = \sum_{i=1}^{n-1}(q^i - 1)$. The proof involves a general property of Tits buildings and a computation of the invariants in $S(V)$ of the parabolic subgroups of $\mathrm{GL}_n \mathbf{F}_q$.

Let $S(V)$ denote the symmetric algebra on the standard n-dimensional representation V of the general linear group $G \equiv \mathrm{GL}_n \mathbf{F}_q$; $\mathbf{F}_q$ is the finite field with $q = p^d$ elements, where p is a fixed prime. In this paper we determine the *multiplicity series* for the Steinberg representation St of G in $S(V)$. This series is defined by

$$F_{\mathrm{St}}(t) = \sum_{k=0}^{\infty} a_k t^k,$$

where a_k is the multiplicity of St as a composition factor in the kth symmetric power of V. Since St is absolutely irreducible and projective, we also have

$$a_k = \dim \hom_G(\mathrm{St}, S^k(V)) = \dim eS^k(V),$$

where $e \in \mathbf{F}_q[G]$ is any idempotent representing St.

THEOREM A.

$$F_{\mathrm{St}}(t) = t^r \prod_{i=1}^{n} \left(1 - t^{q^i - 1}\right)^{-1},$$

where

$$r = \sum_{i=1}^{n-1} (q^i - 1) = \left(\frac{q^n - 1}{q - 1}\right) - n.$$

When $d = 1$, this theorem was first proved in **[6]**, with specific topological applications in mind. When $p = 2$, for example, $S(V)$ can be identified with the cohomology of the classifying space $B\mathbf{F}_2^n$ and, thus, is a graded module over the Steenrod algebra A. Topological constructions had led to the study of a specific

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A-module $M(n)$ whose Poincaré series was identical to the series above, and a small amount of evidence suggested that $M(n)$ was a direct factor in $S(V)$. This was confirmed in **[6]**, where it was shown that $M(n) \simeq eS(V)$ as A-modules.

Other proofs (still with $d = 1$) and more applications appear in **[3, 4, 5]**. All of these approaches make use of the Steenrod algebra to obtain a lower bound on $F_{\mathrm{St}}(t)$.

Now from a topological point of view, the connections with the Steenrod algebra are precisely what make Theorem A so interesting. Nevertheless, the authors have felt for some time that it would be desirable to have a more direct proof of Theorem A that follows classical algebraic lines and is valid for all q. Thus, our method here is to use the theorem of Solomon **[8]**, identifying St with the top homology group of the Tits building Δ.

Solomon's result implies an earlier theorem of Curtis–namely, that in the representation ring of G, $[\mathrm{St}] = \Sigma_I(-1)^{|I|}[1_{P_I}^G]$ (cf. §1; the P_I are the parabolic subgroups of G). In characteristic zero it follows at once that

$$(*) \qquad \text{for any } G\text{-module } M, \quad \dim \hom_G(\mathrm{St}, M) = \sum_I (-1)^{|I|} \dim M^{P_I}.$$

In characteristic $p > 0$, $(*)$ does not follow from Curtis's theorem alone. However, in §1 we observe:

(1.1) THEOREM. *Let G be any finite group with split Tits system of characteristic p. Then the inclusion* St $\to C_*(\Delta)$ *is a* $\mathbf{Z}_{(p)}[G]$*-equivariant chain homotopy equivalence.*

(1.3) COROLLARY. $(*)$ *holds in characteristic p.*

(Theorem (1.1) is presumably known, but does not seem to appear in the literature.) In §2 we then return to the case $G = \mathrm{GL}_n \mathbf{F}_q$ and compute the invariant rings $S(V)^{P_I}$ (Theorem 2.2). (We are told that Theorem 2.2 was known to Mui.) This computation should be of some independent interest, particularly to topologists. (The ring of "Dickson invariants" S^G has been a popular object of study by topologists for some years now; the rings S^{P_I} are of interest in connection with the work of the first author **[4]**.) The proof of Theorem A is then completed by adding up, with signs, the Poincaré series of the rings $S(V)^{P_I}$ (Theorem 2.4). This step is essentially combinatoric; we are greatful to Phil Hanlon for providing the original argument and to the referee for subsequent simplifications.

Finally, it is interesting to note that all the results of this paper can be regarded as analogues of classical theorems concerning the symmetric groups (Remark 2.11).

1. An equivariant form of Solomon's theorem. Let (G, B, N, S) be a (finite) split Tits system of characteristic p and rank r, with Tits building Δ and Weyl group W. For $I \subset S$ there is a parabolic subgroup P_I. Recall that the set of cells of Δ is the set $\bigcup_{I \neq S} G/P_I$, where a cell gP_I is of dimension $r - 1 - |I|$, and the set of faces of $gP_I = \{gP_J | I \subset J\}$. Thus Δ is an $(r-1)$-dimensional G-simplicial complex.

Let $C_*(\Delta)$ be the associated augmented $\mathbf{Z}_{(p)}[G]$ chain complex. Let $\mathrm{St}_* \subset C_{r-1}(\Delta)$ be the module of cycles, viewed as a $\mathbf{Z}_{(p)}[G]$ chain complex concentrated in dimension $r - 1$.

It is well known [8] that the geometric realization of Δ is homotopic to a bouquet of $r-1$ spheres. This implies that the inclusion $i\colon \mathrm{St}_* \hookrightarrow C_*(\Delta)$ is a (nonequivariant) chain homotopy equivalence. Here we note that, because we have localized at p, we have a sharper result.

(1.1) THEOREM. *$i\colon \mathrm{St}_* \hookrightarrow C_*(\Delta)$ is a $\mathbf{Z}_{(p)}[G]$ chain homotopy equivalence.*

This has the following consequences. For $\mathbf{Z}_{(p)}[G]$-modules M_1, M_2, M_3, we write $M_1 = M_2 - M_3$ if $M_1 \oplus M_3 = M_2$.

(1.2) COROLLARY. $\mathrm{St} = \Sigma_I(-1)^{|I|}1_{P_I}^G$ *as* $\mathbf{Z}_{(p)}[G]$*-modules.*

This follows from the theorem by considering Euler characteristics and noting that $C_i(\Delta) = \Sigma_{|I|=r-1-i}1_{P_I}^G$.

Note that the equation $[\mathrm{St}] = \Sigma_I(-1)^{|I|}[1_{P_I}^G]$, valid in the representation ring $R(G)$, holds by the nonequivariant result. Our point is that in spite of the many nontrivial extensions present in the modules $1_{P_I}^G$, (1.2) actually holds as $\mathbf{Z}_{(p)}[G]$-modules.

(1.3) COROLLARY. *If M is an $\mathbf{F}[G]$-module and* $\operatorname{char}\mathbf{F} = p$, *then*

$$\dim \hom_G(\mathrm{St}, M) = \sum_I (-1)^{|I|}\dim M^{P_I}.$$

To prove the theorem, we use the next lemma.

(1.4) LEMMA. *Let G be a finite group and $f\colon C_* \to C'_*$ a chain map between $\mathbf{Z}_{(p)}[G]$ chain complexes. Let $B \subset G$ be a subgroup of index prime to p. If f is a $\mathbf{Z}_{(p)}[B]$ chain homotopy equivalence, then it is a $\mathbf{Z}_{(p)}[G]$ chain homotopy equivalence.*

PROOF. By considering the mapping cone of f, we can assume that $C'_* = 0$. Thus C_* is a $\mathbf{Z}_{(p)}[G]$ chain complex which is contractible as a $\mathbf{Z}_{(p)}[B]$ complex; i.e., C_* is a chain complex formed by splicing together short exact sequences of $\mathbf{Z}_{(p)}[G]$-modules

$$(*) \qquad 0 \to M_1 \to M_2 \to M_3 \to 0$$

which split as sequences of $\mathbf{Z}_{(p)}[B]$-modules. But by the usual "averaging" (transfer) arguments, if there exists a B-equivariant splitting of $(*)$, there exists a G-equivariant one, since $[G : B]$ is prime to p.

By the lemma, to prove the theorem it suffices to show that $i\colon \mathrm{St}_* \hookrightarrow C_*(\Delta)$ is a $\mathbf{Z}_{(p)}[B]$ chain homotopy equivalence. In fact, this is essentially what is shown in the standard proof of the nonequivariant result. We sketch the argument.

Let $l\colon W \to \mathbf{N}$ be the length function, and let $m = l(w_0)$, where w_0 is the longest word element. Δ is filtered by subcomplexes $\Delta_i = \{bwP_I | b \in B \text{ and } l(w) \leqslant i\}$ for $i = 0, \ldots, m$. Note that Δ_i, though not a G complex, *is* a B complex.

In [2, Appendix II] the nonequivariant result follows from the observation that, for $0 < i < m$, Δ_{i-1} is a deformation retract of Δ_i. An inspection of the proof there reveals that the homotopy showing this can be chosen to be B-equivariant: The deformation is obtained via a sequence of elementary collapses, while B merely permutes the cells.

Noting that Δ_0 is the standard $r-1$ simplex with trivial B action, it follows that Δ_i is a B-contractible for $i < m$. Thus π: $C_*(\Delta) \to C_*(\Delta)/C_*(\Delta_{m-1})$ is a $\mathbf{Z}_{(p)}[B]$ chain homotopy equivalence such that πi: $\mathrm{St}_* \to C_*(\Delta)/C_*(\Delta_{m-1})$ is an isomorphism.

2. The multiplicity of St **in** $S(V)$. In this section we set $G = \mathrm{GL}_n \mathbf{F}_q$, where n is a fixed positive integer and $q = p^d$ is a fixed prime power. Parabolic subgroups will be indexed as follows: If $I = (r_1, \dots, r_m)$ is an ordered partition of n, P_I denotes the corresponding standard parabolic subgroup of G with Levi factor $\prod_{i=1}^m \mathrm{GL}_{r_i} \mathbf{F}_q$. Let s_i denote the partial sum $r_1 + \cdots + r_i$ $(1 \leqslant i \leqslant m)$. Then I is uniquely determined by the sequence $(s_1, \dots, s_m)$. We will, ambiguously, let I denote either of these sequences; which of the two is intended will be clear from the context. (Note that I is interpreted in yet another way in §1!) If $S(V)$ is the symmetric algebra on the standard n-dimensional representation of G, our goal is to determine the multiplicity series $F_{\mathrm{St}}(t)$ of St in $S(V)$ (Theorem A), as explained in the introduction. In view of Corollary (1.3), our first task is to calculate the rings of invariants S^{P_I}. The case $P_I = G$ was considered by Dickson [**1**], who proved:

(2.1) THEOREM. *$S(V)^G$ is a polynomial algebra on homogeneous generators $D_{n,k}$ of degree $q^n - q^{n-k}$ $(1 \leqslant k \leqslant n)$. Moreover, these generators satisfy the fundamental equation*

$$\prod_{v \in V} (X - v) = X^{q^n} + \sum_{k=0}^{n-1} D_{n,n-k} X^{q^k}. \qquad \square$$

For a proof, see [**10**]. (Note that Wilkerson's $C_{n,k}$ is our $D_{n,n-k}$, up to a sign which, for our purposes, is irrelevant.) The $D_{n,k}$ are in fact defined via the fundamental equation.

The P_I determine a complete flag $V^1 \subset V^2 \subset \cdots \subset V^n = V$. If $k \leqslant s \leqslant n$, we then regard $D_{s,k}$ as an element of $S(V)$ in the obvious way.

(2.2) THEOREM. *$S(V)^{P_I}$ is a polynomial algebra on the generators $D_{s_i,j}$* $(1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant r_i)$.

REMARK. When $I = (1, 1, \dots, 1)$, so $P^I = B$, these generators are *not* the generators v_i constructed by Mui [7] (except for $D_{1,1} = v_1$).

(2.3) LEMMA. *Suppose R is a subalgebra of $S(V)^H$ (H any subgroup of G) generated by homogeneous elements z_i of degree d_i $(1 \leqslant i \leqslant n)$. Then $R = S(V)^H$ if and only if the following conditions hold, in which case $S(V)^H$ is a polynomial algebra on the z_i:* (a) *S is integral over R, and* (b) $\prod_{i=1}^n d_i = |H|$.

The lemma is easily proved by standard methods of commutative algebra—it is a version of the strategy discussed in §3 of [**10**]. Note that, assuming the validity of the fundamental equation, we see that Dickson's theorem (2.1) follows easily from the lemma.

PROOF OF THEOREM (2.2). It is clear that the $D_{s_i,j}$ in the theorem are, in fact, P_I invariants. Hence it will suffice to verify conditions (a) and (b) of the lemma. The order of P_I is

$$|U_n|\cdot\prod_{i=1}^{m}\left[\mathrm{GL}_{r_i}: U_{r_i}\right] = q^{\binom{n}{2}}\prod_{i=1}^{m}\prod_{j=1}^{r_i}(q^j-1)$$

(U_k is the p-Sylow subgroup of GL_k). On the other hand, the product of the degrees of the $D_{s_i,j}$ is $\prod_{i=1}^m\prod_{j=1}^{r_i}(q^{s_i}-q^{s_i-j})$. A short computation shows that these are equal. We prove condition (b) by induction on the number m of terms in the partition: For $m=1$, (b) follows from (2.1). For the inductive step, let $W=V^{s_{m-1}}$, and consider the extension of algebras $S(W)\to S(V)\to S(V/W)$. Let R_W (resp. R'_W) denote the subalgebra of $S(V)$ generated by the $D_{s_i,j}$ with $i\leqslant m-1$ (resp. by $D_{n,j}$, $1\leqslant j\leqslant r_m$). By induction we can assume $S(W)$ is integral over R_W. Hence, it will be enough, clearly, to show that $S(V/W)$ is integral over R'_W. But it is well known (cf. [**10**, Proposition 1.3b]) that, in general, the image of $D_{n,k}$ in $S(V/W)$ is (up to sign) $D_{t,k}^{q^{n-t}}$ for $1\leqslant k\leqslant t$, where $t=\dim V/W$. Here $t=r_m$, so this completes the proof. □

To complete the proof of Theorem A, we need to establish the following identity:

(2.4) THEOREM. *Let* $r=\sum_{i=1}^{n-1}q^i-1$. *Then*

$$t^r\prod_{i=1}^{n}\left(1-t^{q^i-1}\right)^{-1}=\sum_I(-1)^{|I|}\prod_{i=1}^{m}\prod_{j=1}^{r_i}\left(1-t^{q^{s_i}-q^{s_i-j}}\right)^{-1}.$$

(*Here I is an ordered partition of n as before, with partial sums* s_i, *and* $|I|=n-m$.)

PROOF. Let $x_0,\ldots,x_n$ be indeterminates. We prove the identity

$$(2.5)\quad x_0^{-n}(x_0\cdots x_{n-1})\prod_{i=1}^{n}\left(1-x_0^{-1}x_i\right)^{-1}=\sum_I(-1)^{|I|}\prod_{i=1}^{m}\prod_{j=1}^{r_i}\left(1-x_{s_i-j}^{-1}x_{s_i}\right)^{-1}.$$

(2.4) follows by substituting $x_i=t^{q^i}$. Now observe that the elements x_{s_i-j} occurring on the right-hand side of (2.5) are precisely the variables $x_0,x_1,\ldots,x_{n-1}$ (each of these occurring exactly once). Dividing by $x_0\cdots x_{n-1}$ yields the equivalent equation

$$(2.6)\quad \prod_{i=1}^{n}(x_0-x_i)^{-1}=\sum_I(-1)^{|I|}\prod_{i=1}^{m}\prod_{j=1}^{r_i}\left(x_{(s_i-j)}-x_{s_i}\right)^{-1}.$$

Let $f_n(x_0,x_1,\ldots,x_n)$ denote the right-hand side of (2.6). We have the recursive formula

$$f_n(x_0,\ldots,x_n)=f_{n-1}(x_0,\ldots,x_{n-2},x_n)(x_{n-1}-x_n)^{-1}$$
$$-f_{n-1}(x_0,\ldots,x_{n-1})(x_{n-1}-x_n)^{-1},$$

with the first term being the contribution of those I's with $r_m>1$, and the second, of those with $r_m=1$. From this and $f_1(x_0,x_1)=(x_0-x_1)^{-1}$, (2.6) follows by induction on n.

(2.11) REMARK. We comment on a well-known analogue of our results. Suppose we replace $GL_n F_q$ by the symmetric group Σ_n. The analogue of the Steinberg module is the sign representation, and the analogue of the Tits complex is the Coxeter complex of chains of proper nonempty subsets of $\{1, \ldots, n\}$ (barycentric subdivision of the standard $(n-2)$-sphere). The parabolic subgroups P_I are then just the subgroups $\prod_{i=1}^{m} \Sigma_{r_i}$. Over a field of characteristic zero we then obtain the following well-known identities, where $F_{\mathrm{sgn}}(t)$ is the multiplicity series for sgn in $S(V)$:

$$(2.12) \qquad F_{\mathrm{sgn}(t)} = \sum_I (-1)^{|I|} \prod_{i=1}^{m} \prod_{j=1}^{r_i} (1 - t^j)^{-1} = t^{\binom{n}{2}} \prod_{i=1}^{n} (1 - t^i)^{-1}.$$

Here the first equality follows from the analogues of (1.3) and (2.2), and the second is the specialization of (2.5) obtained by setting $x_i = t^i$. (Of course $F_{\mathrm{sgn}}(t)$ can also be easily computed by other means.)

REMARK. Theorem A can be generalized in two ways; these generalizations (with $d = 1$) were essential in the topological applications [**6**, **5**]. The first is to compute the multiplicity series for St in $S(V) \otimes E(V)$, where $E(V)$ is the exterior algebra on V. Here it would be advisable to use the results rather than the methods of this paper, since even the B-invariants of $S \otimes E$ are rather complicated [**7**]. The second is to compute the corresponding series for $\det^k \otimes \mathrm{St}$, where det is the determinant representation and $0 \leqslant k \leqslant q - 2$. We will not compute any of these series here, but we remark that it would not be hard to do so by combining Theorem A with the methods of [**5**, §1].

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