THE DISTANCE BETWEEN THE EIGENVALUES OF HERMITIAN MATRICES

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ABSTRACT. It is shown that the minmax principle of Ky Fan leads to a quick simple derivation of a recent inequality of V. S. Sunder giving a lower bound for the spectral distance between two Hermitian matrices. This brings out a striking parallel between this result and an earlier known upper bound for the spectral distance due to L. Mirsky.

Let \( A \) be a Hermitian matrix of order \( n \) and let \( \lambda_1(A) \) denote the vector in \( \mathbb{R}^n \) whose coordinates are the eigenvalues of \( A \) arranged as \( \lambda_1(A) \geq \cdots \geq \lambda_n(A) \). Let \( \lambda_{(1)}(A) \leq \cdots \leq \lambda_{(n)}(A) \) be the increasing rearrangement of these eigenvalues and \( \lambda_j(A) \) the vector with coordinates \( \lambda_{(j)}(A) \), \( j = 1, 2, \ldots, n \). The same symbols \( \lambda_j(A) \) and \( \lambda_j(A) \) will also denote the diagonal matrices which have as their diagonal entries the components of the vectors \( \lambda_j(A) \) and \( \lambda_j(A) \), respectively. Let \( \| \cdot \| \) denote any unitarily invariant norm on the space of matrices. (See [4].)

This note is concerned with the following result:

**THEOREM.** Let \( A \) and \( B \) be Hermitian matrices. Then for every unitarily invariant norm we have

\[
\| \lambda_1(A) - \lambda_1(B) \| \leq \| A - B \| \leq \| \lambda_1(A) - \lambda_1(B) \|.
\]

The first inequality in (1) appeared in a paper of Mirsky [4], who used a famous result of Lidskii and Wielandt to derive it. The second is proved in a recent paper of Sunder [5]. I give here another proof of the second inequality which has two attractive features: It is very short and it proceeds on exactly the same lines as the well-known proof of Lidskii, Wielandt and Mirsky for the first inequality. For illumination, I indicate how both inequalities follow from the same principle.

It is an easy consequence of the minmax principle of Wielandt that for any choice \( 1 \leq i_1 < \cdots < i_k \leq n \) of \( k \) indices we have

\[
\sum_{j=1}^{k} \lambda_{[i_j]}(A + B) \leq \sum_{j=1}^{k} \lambda_{[j]}(A) + \sum_{j=1}^{k} \lambda_{[i_j]}(B)
\]

for all \( k = 1, 2, \ldots, n \), with equality holding for \( k = n \). (See [3, p. 242].)

Writing \( x \prec y \) to mean that the vector \( x \) is majorised by the vector \( y \) in \( \mathbb{R}^n \) (see [3]), we get from inequalities (2)

\[
\lambda_1(A + B) - \lambda_1(B) < \lambda_1(A).
\]

With a change of variables, this gives

\[
\lambda_1(A) - \lambda_1(B) < \lambda_1(A - B).
\]
Now the first part of the Theorem follows using standard characterisations of majorisation together with properties of symmetric gauge functions and unitarily invariant norms. This is the well-known proof of Mirsky [4].

Now note that from (2) we can also conclude
\[ \lambda_1(A + B) \prec \lambda_1(A) + \lambda_1(B). \]
In fact, for this conclusion the full force of (2) is not needed. It suffices to use the special case \((i_1, \ldots, i_k) = (1, \ldots, k)\) which is much easier to prove using the minmax principle of Ky Fan [2].

Replace \(B\) by \(-B\) in (4) and note that \(\lambda_1(-B) = -\lambda_1(B)\). This gives
\[ \lambda_1(A - B) \prec \lambda_1(A) - \lambda_1(B). \]
But this implies
\[ (|\lambda_1(A - B)|, \ldots, |\lambda_n(A - B)|) \prec_w (|\lambda_1(A) - \lambda_1(B)|, \ldots, |\lambda_n(A) - \lambda_n(B)|) \]
where \(\prec_w\) stands for weak majorisation [3, p. 116].

Let \(s_{[j]}(A)\) denote the \(j\)th singular value of \(A\). Let \(\|A\|_k = s_{[1]}(A) + \cdots + s_{[k]}(A)\) for \(k = 1, 2, \ldots, n\). Then (5) can be restated as \(\|A - B\|_k \leq \|\lambda_1(A) - \lambda_1(B)\|_k\), \(k = 1, 2, \ldots, n\). So the second inequality in (1) holds for this special class of norms and hence, by a well-known theorem of Ky Fan, for every unitarily invariant norm. (See [4].)

It should be remarked that Sunder’s paper contains a stronger result in that it also establishes an analogue of the second inequality in (1) for the case when \(A, B\) and \(A - B\) are all normal. Under these conditions an analogue of the first inequality in (1) has been established in [1].

REFERENCES

5. V. S. Sunder, On permutations, convex hulls and normal operators, Linear Algebra and Appl. 48 (1982), 403-411.

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